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ON THE THEORY OF AUTOMISERS IN FINITE GROUPS

By

M. NIEMENMAA

Department of Mathematics, University of Oulu, Finland

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Let G be a finite group and H a subgroup of G. We say that the factor group $N_G(H)/C_G(H)$ is the automiser of H in G. We define

$$a_G(H) = N_G(H)/C_G(H).$$

If $N_G(H) = C_G(H)$, then we write $a_G(H) = 1$ and we say that $a_G(H)$ is trivial. The concept of an automiser is introduced in [1] (p. 596), where a class of simple groups is characterized with the aid of automisers.

In the present paper we show that this concept is useful in the case of non-simple groups, too. We study the effect on a finite group G of certain conditions imposed on the automisers of the subgroups of G.

Our notation is generally the same as in [3]. However, we write $O(G)$ to mean the order of a group G. We also write $N(H)$, $C(H)$ and $a(H)$ for $N_G(H)$, $C_G(H)$ and $a_G(H)$ if there is no danger of confusion.

We first introduce the following classical result which describes the structure of minimal non-abelian groups (that is, non-abelian groups which have only abelian subgroups). We call these groups Rédei groups.

THEOREM 1. *Let G be a non-primary Rédei group, then $G = PQ$, where P is an elementary abelian minimal normal Sylow subgroup of G and Q is a cyclic non-normal Sylow subgroup of G.*

For the proof, see [5]. For a modern treatment, see [4], chapter III, section 5.

If $a(H) = 1$ for all the proper subgroups H of G, then, clearly, G is abelian. The following theorem shows us the effect of cyclic automisers on G.

THEOREM 2. *Let G be a non-abelian group such that $a(H)$ is cyclic for all the proper subgroups H of G. Then G is a Rédei group and conversely.*

PROOF. Suppose that G is non-abelian and $a(H)$ is cyclic for all $H < G$. Let A be a maximal subroup of G. Now $A/Z(A)$ is cyclic, so A is abelian. Thus all the proper subroups of G are abelian.

Then suppose that G is a Rédei group. Let G be primary, that is, $O(G) = p^n$. If $H < G$, then H is contained in a maximal subgroup A of G. Since $N(H) \geq C(H) \geq A$, we conclude that $a(H)$ is cyclic. If is G non-primary, then G has the structure described in theorem 1. The non-normal Sylow subgroup Q is maximal in G, so $N(Q) = C(Q) = Q$. Furthermore, if $K < Q$, then $K \leq Z(G)$. Now it is easy to see that $a(H)$ is cyclic for all $H < G$.

In the case of non-primary Rédei groups we gain a slight refinement of theorem 2 by proving

THEOREM 2′. *Let G be a non-nilpotent group. If $a(H)$ is nilpotent for all the proper subgroups H of G, then G is a Rédei group.*

PROOF. Suppose that G is non-nilpotent and let $a(H)$ be nilpotent for all $H < G$. If A is a maximal subgroup of G, then $A/Z(A)$ is nilpotent, hence A is nilpotent, too. Thus all the proper subgroups of G are nilpotent.

Since G is non-nilpotent, it follows that $G = PQ$, where P is a normal Sylow subgroup of G and Q is a non-normal cyclic Sylow subgroup of G. Furthermore, the maximal subgroup D of Q is contained in $Z(G)$, (see [4], p. 281).

Now Q is not contained in $C_G(P)$. Thus

$$C_G(P) = Z(P)\, D\,.$$

Suppose that $Z(P) < P$. Then $Z(P)Q$ is nilpotent and thus all the elements of $Z(P)$ commute with all the elements of Q. Thus $Z(P) < Z(G)$, hence $Z(P)D \leq Z(G)$. On the other hand,

$$C_G(P) \geq C_G(G) = Z(G)\,,$$

so $C_G(P) = Z(G)$. Now

$$N_G(P)/C_G(P) = G/Z(G)$$

is nilpotent, so G is nilpotent, too. This is a contradiction and we conclude that $Z(P) = P$. Thus all the proper subgroups of G are abelian and G is a Rédei group.

By theorems 2 and 2′ it is easy to see that if G is non-nilpotent and $a(H)$ is nilpotent for all $H < G$, then, in fact, $a(H)$ is cyclic for all $H < G$.

We shall now study the structure of groups G in which the proper subgroups are either normal in G or have trivial automisers in G but can not have both properties. We call these groups T-groups. As an obvious consequence of the definition of T-groups we conclude that $Z(G) = 1$ for all T-groups G.

The following theorem characterizes all T-groups. In the proof we use the properties of Frobenius groups (see [4], pp. 495–508).

THEOREM 3. *Let G be a T-group. Then G is a Frobenius group with kernel P and cyclic complement M. Furthermore, $N(H) = P$ for all $1 < H < P$. Conversely, a Frobenius group with these properties is a T-group.*

PROOF. Suppose that G is a T-group. Since G is non-nilpotent, it follows that G has a maximal subgroup M such that $N(M) = C(M) = M$. Furthermore, $C_G(x) \leq M$ for all the non-unit elements of M. Thus G is a Frobenius group with complement M. Now M is abelian, so all the Sylow subgroups of M are cyclic, hence M is cyclic, too. Let P be the Frobenius kernel of G. From the maximality of M it follows that P is a minimal normal subgroup of G. Clearly, G is solvable and thus P is elementary abelian. Let $1 < H < P$. Now H has a trivial automiser in G, so $N(H) = C(H) = P$.

Now suppose that G is a Frobenius group possessing the properties given in the theorem. Now $N(M) = C(M) = M$ and $N(K) = C(K) = M$ for all $1 < K < M$. Thus all subgroups of M have trivial automisers in G. As in the first part of the proof, we conclude that P is elementary abelian and thus all proper subgroups of P have trivial automisers in G. Subgroups of the form PK, where K is contained in a conjugate of M, are normal in G. Suppose there exists a subgroup of the form HK, where $1 < H < P$ and $1 < K$ is contained in a conjugate of M. Then HK is a Frobenius group with kernel H and complement K, so that $N(H) > K$, a contradiction. Thus we have shown that G is a T-group.

Let $G = P\langle h\rangle$ be a T-group, $O(P) = p^k$ and $O(h) = n$. We consider P as a k-dimensional vector space over $GF(p)$. Let X be the linear transformation induced by h. Then $X^{n-1} + \ldots + X + I = 0$ and by theorem 3, the only non-trivial X-invariant subspace of P is P itself. Now we can state a corollary on the structure of T-groups.

Corollary. Now $GF(p^k)$ is the splitting field for $f(x) = x^{n-1} + \ldots + x + 1$ over $GF(p)$, P is the additive group of $GF(p^k)$ and $g^h = ag$ for all $g \in P$, $a \in GF(p^k)$ and $f(a) = 0$. If $k > 1$, then h is of prime order q and $k = q-1$.

The proof follows from theorem 3.10 of [4], pp. 165–166 and from the fact that if $k > 1$, then $f(x)$ is irreducible over $GF(p)$.

We remark that if $O(h)$ is even or if $O(h)$ is divisible by 3 and $p = 6n+1$, then $k = 1$.

We also remark that T-groups belong to the class of groups characterized by Csörgő [2].

Now it is natural to ask how the structure is changed if in the definition of T-group the condition "$a(H) = 1$ for every non-normal subgroup H of G" is replaced by the condition "$a(H)$ is cyclic for every non-normal subgroup H of G".

It is easy to see that in studying the structure of such groups we can proceed in the same way as in the proof of theorem 3 and we get the same structure as in theorem 3. But now the Frobenius kernel P has a cyclic automiser in G, a contradiction. Thus there exist no groups which satisfy the "new" definition.

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ОТКРЫТЫЕ ОТОБРАЖЕНИЯ ПОЛУСИНТОПОГЕННЫХ ПРОСТРАНСТВ

МАСЛЮВ В. В.
Ворошиловград

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Как известно, топологические, близостные и равномерные структуры являются весьмя частными случаями так называемых синтопогенных структур, первоначальное изучение которых проведено А. Часаром (см. [2]). Понятие «полусинтопогенная структура» есть обобщение понятия «синтопогенная структура»; полусинтопогенные структуры впервые рассматривались в ([1]). С терминологией можно ознакомиться в указанных работах.

В настоящей статье изучаются некоторые отображения полусинтопогенных пространств, названные «открытыми».

Определение 1. Отображение $f:[X_1; S_1]\to[X_2; S_2]$ будем называть $(S_1; S_2)$-открытым, если каждому полутопогенному порядку $<_1\in S_1$ соответствует такой полутопогенный порядок $<_2\in S_2$, что из каждого соотношения $A<_1 B$ вытекает соотношение $f(A)<_2 f(B)$.

Замечания. 1. Если $[X_1; \{<_1\}]$ и $[X_2; \{<_2\}]$-топологические пространства, то понятие $(\{<_1\}; \{<_2\})$-открытости отображения $f: X_1\to X_2$ является ни чем иным, как понятием открытости (в обычном топологическом смысле).

2. Пусть $[X_1; \{<_1\}]$ и $[X_2; \{<_2\}]$-пространства близости, а отображение $f: X_1\to X_2$ является $(\{<_1\}; \{<_2\})$-открытым. Тогда каждое соотношение $A<_1 B$ влечет соотношение $f(A)<_2 f(B)$. Пусть δ_i, $i=1, 2$-классическое отношение близости, соответствующее структуре $\{<_i\}$, $i=1, 2$, а $\bar{\delta}_i$, $i=1, 2$-отрицание этого отношения. В таком случае, как известно, соотношение $A<_1 B$ означает, что $A\,\bar{\delta}_1\, X_1\setminus B$ (соответственно, $f(A)\,\bar{\delta}_2\cdot$ $\cdot X_2\setminus f(B)$), и обратно. Иными словами, отображение $f:[X_1, \delta_1]\to[X_2, \delta_2]$ является открытым тогда и только тогда, когда выполняется следующее условие: «множества $f(A)$ и $X_2\setminus f(B)$ δ_2-далеки каждый раз, когда δ_1-далеки множества A и $X_1\setminus B$».

3. Допустим сейчас, что $[X_1; S_1]$ и $[X_2; S_2]$-равномерные пространства, а $f: X_1\to X_2$ – $(S_1; S_2)$-открытое иньективное отображение. Пусть

порядок $<_2 \in S_2$ соответствует порядку $<_1 \in S_1$ так, как того требует определение 1. Порядку $<_i$, $i = 1, 2$, сопоставляется окружение U_i, $i = 1, 2$, ассоциированной классической равномерной структуры на X_i, $i = 1, 2$, правилом: $(x; y) \in U_i$ тогда и только тогда, когда не верно соотношение $\{x\} <_i X_i \setminus \{y\}$, $i = 1, 2$. Взяв отрицание, найдем: $(x; y) \notin U_i$ тогда и только тогда, когда $\{x\} <_i X_i \setminus \{y\}$, $i = 1, 2$. Итак, по условию: $\{x\} <_1 X_1 \setminus \{y\} \Rightarrow \Rightarrow \{f(x)\} <_2 f(X_1 \setminus \{y\}) \subset X_2 \setminus \{f(y)\}$. Иначе говоря, инъективное отображение f является $(S_1; S_2)$-открытым тогда и только тогда, когда выполняется следующее условие: «каждому окружению U_1 равномерной структуры, ассоциированной со структурой S_1, соответствует окружение U_2 равномерной структуры, ассоциированной со структурой S_2, так, что каковы бы ни были U_1-далекие элементы $(x, y) \in X_1 \times X_1$, образы этих элементов $\big($т. е. $f(x), f(y)\big)$ являются U_2-далекими».

Предложение 1. Если отображение $f: [X_1; S_2] \to [X_2; S_2]$ является $(S_1; S_2)$-открытым, то для каждой полусинтопогенной структуры S_1' (соответственно, S_2') на классе X_1, более слабой (соответственно, сильной), чем структура S_1 (соответственно, S_2), отображение f является $(S_1'; S_2')$-открытым. Доказательство элементарно.

Следствия. 1. Пусть отображение $f: [X_1; S_1] \to [X_2; S_2]$ является $(S_1; S_2)$-окрытым; тогда это отображение будет и $(S_1; S_2^q)$, $(S_1; S_2^p)$, (S_1, S_2^t), $(S_1; S_2^s)$-открытым.

2. Если отображение f из равномерного пространства $[X_1; S_1]$ в равномерное пространство $[X_2; S_2]$ является $(S_1; S_2)$-открытым, то f является $(S_1; S_2^{tq})$-(соответственно, $(S_1; S_2^{tqp})$-) открытым отображением равномерного пространства $[X_1; S_1]$ в пространство близости $[X_2; S_2^{tq}]$ (соответственно, в топологическое пространство $[X_2; S_2^{tqp}]$).

3. Если отображение f из пространства близости $[X_1; \{<_1\}]$ в пространство близости $[X_2; \{<_2\}]$ является $(\{<_1\}, \{<_2\})$-открытым, то f — $(\{<_1\}, \{<_2^p\})$-открыто (как отображение из пространства близости $[X_1; \{<_1\}]$ в топологическое пространство $[X_2; \{<_2^p\}]$).

Предложение 2. Если отображение $f: [X_1; S_1] \to [X_2; S_2]$ — $(S_1; S_2)$-открыто, то оно является и $(S_1^p; S_2^p)$-открытым.

Доказательство. Совершенный полутопогенный порядок $<_1^p \in S_1^p$ определяется полутопогенным порядком $<_1 \in S_1$, для которого можно, пользуясь определением 1, найти полутопогенный порядок $<_2 \in S_2$, и тем самым, — совершенный полутопогенный порядок $<_2^p \in S_2^p$. Покажем, что порядок $<_2^p$ удовлетворяет определению 1.

Пусть $A <_1^p B$; это значит, что найдутся такой класс индексов I и для каждого $i \in I$ — такая часть A_i класса X_1, что $A = \bigcup_{i \in I} A_i$ и $A_i <_1 B$ при всех $i \in I$. Тогда: $f(A_i) <_2 f(B_i)$ при $i \in I$, и $\bigcup_{i \in I} f(A_i) <_2^p f(B)$. Так как $\bigcup_{i \in I} f(A_i) = f\Big(\bigcup_{i \in I} A_i\Big) = f(A)$, то $f(A) <_2^p f(B)$.

Следствие. Если $f: [X_1; \{<_1\}] \to [X_2; \{<_2\}]$ является $(\{<_1\}; \{<_2\})$-открытым отображением пространства близости $[X_1; \{<_1\}]$ в пространство близости $[X_2; \{<_2\}]$, то f — $(\{<_1^p\}; \{<_2^p\})$-открыто как отображение топологического пространства $[X_1; \{<_1^p\}]$, выводимого из пространства близости $[X_1; \{<_1\}]$, в топологическое пространство $[X_2; \{<_2^p\}]$, выводимое из пространства близости $[X_2; \{<_2\}]$.

Предложение 3. Если отображение $f: [X_2; S_1] \to [X_2; S_2] - (S_1; S_2)$-открыто, то оно является $(S_1^l; S_2^l)$-открытым.

Доказательство. Положим $S_i^l = \{<_i\}$, $i = 1, 2$. Если $A <_1 B$, то $A < B$ при некотором $< \in S$, так что $f(A) <' f(B)$ (мы считаем, что порядок $<' \in S_2$ удовлетворяет определению 1), а потому: $f(A) <_2 f(B)$.

Предложение 4. Если инъективное отображение $f: X_1 \to X_2$ $(S_1; S_2)$-открыто, то оно является $(S_1^q; S_2^q)$-открытым.

Доказательство. Каждый топогенный порядок $<_1^q$ из синтопогенной структуры S_1^q определяется полутопогенным порядком $<_1$ из полусинтопогенной структуры S_1, для которого в полусинтопогенной структуре S_2 существует полутопогенный порядок $<_2$, удовлетворяющий определению 1. Покажем, что порядок $<_2^q \in S_2^q$-искомый.

Пусть $A <_1^q B$; это значит, что найдутся такие натуральные числа m, n, и части A_i, $i = 1, 2, \ldots, m$, B_j, $j = 1, 2, \ldots, n$, класса X_1, что: $A = \bigcup_{i=1}^{m} A_i$, $B = \bigcap_{j=1}^{n} B_j$ и $A_i <_1 B_j$ при всех значениях индексов i и j.

Тогда $f(A_i) <_2 f(B_j)$ и $\bigcup_{i=1}^{m} f(A_i) <_2^q \bigcap_{j=1}^{n} f(B_j)$. Поскольку

$$f(A) = f\left(\bigcup_{i=1}^{m} A_i\right) = \bigcup_{i=1}^{m} f(A_i) \text{ и } f(B) = f\left(\bigcap_{j=1}^{n} B_j\right) = \bigcap_{j=1}^{n} f(B_j)$$

(здесь используется инъективность), то $f(A) <_2^q f(B)$.

Следствия. 1. Если инъективное отображение $f: [X_1; S_1] \to [X_2; S_2] - (S_1; S_2)$-открыто, то оно является и $(S_1^s; S_2^s)$-открытым.

Для доказательства достаточно заметить, что каждый порядок структуры S^s имеет вид $(< \cup <^l)^q$, где $< \in S$.

2. Если отображение $f: [X_1; S_1] \to [X_2; S_2]$ является инъективным и $(S_1; S_2)$-открытым, то это отображение является и $(S_1^{lq}; S_2^{lq})$-открытым.

Заметим, что если считать, в частности, что S_i, $i = 1, 2$, — равномерные структуры, то S_2^{lq}, $i = 1, 2$, — соответствующие структуры близости.

Примеры. 1. Пусть S — полусинтопогенная структура на классе $X \neq \emptyset$, каждый полутопогенный порядок которой «замкнут» относительно конечных пересечений (но не является, вообще говоря, топогенным, так что S — вобщем-то, не синтопогенная структура), а $A \neq \emptyset$ — произвольная часть класса X. Для того, чтобы каноническая инъекция $i: A \to X$ была $(S|A; S)$-открытой, необходимо и достаточно, чтобы класс A был $<$-открыт при всех $< \in S$.

2. Для того, чтобы биекция $f: [X_1; S_1] \to [X_2; S_2]$ была изоморфизмом, необходимо и достаточно, чтобы отображение f было $(S_1; S_2)$-непрерывным и $(S_1; S_2)$-открытым.

3. Если наделить класс $X_1 \neq \emptyset$ структурой $f^{-1}(S_2) \overset{\text{def}}{=} S_1$, где f — сюръекция класса, X_1 на полусинтопогенное пространство $[X_2; S_2]$, то f окажется $(S_1; S_2)$-открытым отображением.

4. Каждая проекция pr_i, $i \in I$, произведения $\left[\prod_{i \in I} X_i; \prod_{i \in I} S_i\right]$ полусинтопогенных пространств $[X_i; S_i]$, $i \in I$, на пространство $[X_i, S_i]$, $i \in I$, является $\left(\prod_{i \in I} S_i; S_i\right)$-открытым отображением.

Предложение 5. Пусть $f : [X; S] \to [X'; S']$ и $g : [X'; S'] \to [X''; S'']$ — два отображения. Верны следующие утверждения:

(1) если отображение $f - (S; S')$-открыто, а $g - (S'; S'')$-открыто, то отображение $g \circ f - (S; S'')$-открыто;

(2) если отображение $h \circ f - (S; S'')$-открыто, а отображение f-сюръективно и $(S; S')$-непрерывно, то отображение $g - (S'; S'')$-открыто;

(3) если отображение $g \circ f - (S; S'')$-открыто, а отображение g-инъективно и $(S'; S'')$-непрерывно, то отображение f является $(S; S')$-открытым.

Доказательство. Утверждение (1) очевидно.

(2) Произвольному полутопогенному порядку $<' \in S'$ соответствует (в силу $(S; S')$-непрерывности отображения f) полутопогенный порядок $< \in S$, а для него — полутопогенный порядок $<'' \in S''$ (в силу $(S; S'')$-открытости отображение $g \circ f$). Если $A <' B$, то $f^{-1}(A) < f^{-1}(B)$ и $g \circ f(f^{-1}(A)) <'' g \circ f(f^{-1}(B))$. Так как f сюръективно, то $f(f^{-1}(A)) = A$ и то $f(f^{-1}(B)) = B$, а потому $g(A) <'' (B)$.

(3) Если $< \in S$, то $(S; S'')$-открытость отображения $g \circ f$ влечет возможность выбора полутопогенного порядка $<'' S_2''$, удовлетворяющего определению I; используя $(S'; S'')$-непрерывность отображения g, найдем полутопогенный порядок $<' \in S'$, мажорирующий полутопогенный порядок $g^{-1}(<'')$. Если $A < B$, то $g \circ f(A) <'' g \circ f(B)$ и $g^{-1}(g \circ f(A)) <' g^{-1}(g \circ f(B))$. Так как g инъективно, то $g^{-1}(g \circ f(A)) = f(A)$ и $g^{-1}(g \circ f(B) = f(B)$, а потому $f(A) <' f(B)$.

Предложение 6. Пусть $f : [X_1; S_1] \to [X_2, S_2]$-произвольное $(S_1; S_2)$-открытое отображение. Для непустой части B класса X_2 символом f_B обозначим отображение из класса $f^{-1}(B)$ в класс B, совпадающее с f на $f^{-1}(B)$. Оказывается, что это отображение является $(S_1 | f^{-1}(B); S_2 | B)$-открытым.

Доказательство. Пусть $<$-произвольный полутопогенный порядок структуры $S_1 | f^{-1}(B)$, определяемый полутопогенным порядком $<_1 \in S_1$. Для $<_1$ найдем полутопогенный порядок $<_2 \in S_2$ так, как того требует определение I. Наконец, положим $<' = <_2 | B$. Если $U < V$, то $U <_1 V \cup \cup (X_1 \setminus f^{-1}(B))$, а потому

$$f(U) <' f(V \cup (X_1 \setminus f^{-1}(B)) = f(V) \cup f(X_1 \setminus f^{-1}(B)) \subset f(V) \cup (X_2 \setminus B),$$

так что $f(U) <' f(V)$.

Предложение 7. Пусть f-некоторое отображение полусинтопогенного пространства $[X; S]$ в синтопогенное пространство $[X'; S']$. Предположим, что выполняются следующие условия:

(1) выделено конечное покрытие (B_i) $i = 1, 2, \ldots, n$, класса X';
(2) все классы B_i удовлетворяют соотношению $B_i < B_i$ при всех $< \in S'$;
(3) все отображения f_{B_i} — $(S|f^{-1}(B_i); S'|B_i)$-открыты. Тогда оказывается, что отображение f является $(S; S')$-открытым.

Доказательство. Выберем произвольно полутопогенный порядок $< \in S$; для $<_i = <|f^{-1}(B_i)$ найдем $<_i'' \in S'|B_i$ так, как того требует определение 1. Если $<_i'' = <_i'|B_i$, $i = 1, 2, \ldots, n$, то пусть $<' \in S'$ — тот топогенный порядок, который мажорирует все порядки $<_i'$, $i = 1, 2, \ldots, n$. Пусть $U < V$; тогда $U \cap f^{-1}(B_i) <_i V \cap f^{-1}(B_i)$ при $i = 1, 2, \ldots, n$, а потому $f_{B_i}(U \cap f^{-1}(B_i)) <_i'' f_{B_i}(V \cap f^{-1}(B_i))$, т. е.

$$f(U) \cap B_i <_i'' f(V) \cap B_i,$$

так что

$$f(U) \cap B_i <' (f(V) \cap B_i) \cup (X' \setminus B_i).$$

Так как $B_i <' B_i$, то

$$(f(U) \cap B_i) \cap B_i <' \big((f(V) \cap B_i) \cup (X' \setminus B_i)\big) \cap B_i,$$

и поэтому

$$f(U) \cap B_i <' f(V) \cap B_i.$$

Отсюда:

$$\bigcup_{i=1}^{n} (f(U) \cap B_i) <' \bigcup_{i=1}^{n} (f(V) \cap B_i);$$

вместе с этим: $f(U) <' f(V)$.

Замечание. Доказанное предложение можно несколько видоизменить, приспособив его к тому случаю, когда покрытие (B_i) класса X' не является конечным. Однако, тогда придется требовать, чтобы структура S' была синтопологией. При этом, в частности, получается известный результат, касающийся топологических пространств; а именно: если $[X; S]$ — полусинтопогенное пространство, $[X'; \{<'\}]$ — топологическое пространство, и семейство $(B_i)_{i \in I \neq \emptyset}$ образует покрытие класса X' $\{<'\}$-окрытыми классами, а все отображения f_{B_i}, $i \in I$, являются $(S|f^{-1}(B_i); \{<'\}|B_i)$-открытыми, то $(S; \{<'\})$-открыто отображение $f: X \to X'$. Отметим еще, что указанный факт до сих пор формулировался лишь в том случае, когда структура S предполагалась топологией.

Предложение 8. Пусть $f_i: [X_i; S_i] \to [Y_i; S_i']$, $i \in I \neq \emptyset$, — $(S_i; S_i')$-открытая сюръекция. Тогда оказывается, что отображение $f = (f_i): \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ является $\left(\prod_{i \in I} S_i; \prod_{i \in I} S_i'\right)$-открытым.

Наметим лишь идею доказательства.

Каждый полутопогенный порядок $< \in \prod_{i \in I} S_i$ определяется конечным семейством полутопогенных порядков $<_\alpha \in S_\alpha$, $\alpha \in A \subset I$; каждому $<_\alpha \in S_\alpha$, $\alpha \in A$, соответствует, согласно определению 1, полутопогенный порядок $<_\alpha' \in S_\alpha'$, $\alpha \in A$. Семейство порядков $(<_\alpha')_{\alpha \in A}$ определяет полутопогенный порядок $<' \in \prod_{i \in I} S_i'$. Именно этот-то порядок и является искомым.

Определение 2. Многозначное отображение $f:[X_1;S_1]\to[X_2;S_2]$ будем называть $(S_1;S_2)$-открытым, если каждому полутопогенному порядку $<_1\in S_1$ соответствует такой полутопогенный порядок $<_2\in S_2$, что каждое соотношение $A<_1 B$ имеет следствием соотношение $f(A)<_2 f(B)$.

Определение 3. Пусть $<$ — некоторый полутопогенный порядок на классе $X\neq\emptyset$. Между подклассами A, B класса $\mathcal{D}(X)$ определим отношение $<_\lambda$ следующим правилом: $A<_\lambda B$ тогда и только тогда, когда: (1) либо $A=B=\emptyset$, (2) либо найдутся такие две части U и V класса X, что $U<V$,

$$A\subset\{(W):W\cap U\neq\emptyset\};\quad \{(U):W\cap V\neq\emptyset\}\subset B.$$

Предложение 9. Отношение $<_\lambda$ является полутопогенным порядком на классе $\mathcal{D}(X)$, сужение которого на X есть $<$.

Предложение 10. Если S — полусинтопогенная структура на классе $X\neq\emptyset$, то класс $S_\lambda=\{<_\lambda:<\in S\}$ есть полусинтопогенная структура на $\mathcal{D}(X)$.

Замечание. Изучение свойств структуры S_λ (равно, как и свойств некоторых других структур на классе $\mathcal{D}(X)$) проводится в одной из работ автора, которая в настоящее время находится в печати.

Предложение 11. Если многозначное отображение $f:[X_1;S_1]\to\to[X_2;S_2]$ является $(S_1;S_2)$-открытым, то однозначное отображение $\hat{f}$, определяемое правилом: $(A)\to(f(A))$, если $(A)\in\mathcal{D}(X)$, является $(S_{1\lambda};S_{2\lambda})$-открытым.

Доказательство. Пусть $<_{1\lambda}\in S_{1\lambda}$ — произвольный полутопогенный порядок, определяемый порядком $<_1\in S_1$. Покажем, что полутопогенный порядок $<_{2\lambda}\in S_{2\lambda}$, определяемый полутопогенным порядком $<_2\in S_2$, соответствующим порядку $<_1$ так, как того требует определение 2, удовлетворяет определению 1.

Если $A<_{1\lambda}B$, то найдутся такие части U, V класса X, что

$$U<_1 V,\quad A\subset\{(W):W\cap U\neq\emptyset\},\quad \{(W):W\cap V\neq\emptyset\}\subset B.$$

Так как

$$f(U)<_2 f(B),\quad\text{то}\quad \{(W):W\cap f(U)\neq\emptyset\}<_{2\lambda}\{(W):W\cap f(V)\neq\emptyset\}.$$

Если $(z)\in\hat{f}(A)$, то найдется такой элемент $(t)\in A$, что $(z)=\hat{f}((t))$. Так как $(t)\in A$, то $t\cap U\neq\emptyset$, так что $f(t\cap U)\neq\emptyset$. Поскольку $f(t)\cap f(U)\supset\supset f(t\cap U)$, то $f(t)\cap f(U)\neq\emptyset$, а потому $(f(t))\in\{(W):W\cap f(U)\neq\emptyset\}$, так что $(z)\in\{(W):W\cap f(U)\neq\emptyset\}$. Таким образом, $\hat{f}(A)\subset\{(W):W\cap f(U)\neq\emptyset\}$. Аналогично можно доказать включение $\{(W):W\cap f(V)\neq\emptyset\}\subset\hat{f}(B)$.

Итак, $\hat{f}(A)<_{2\lambda}\hat{f}(B)$.

Замечание. Частный случай предложения 11, когда $[X_i;S_i]$, $i==1,2$, — топологические пространства, рассматривался Линичуком Р. С.

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A REMARK ON CONTINUOUS INDEPENDENT FUNCTIONS

By

W. BRYC

Institute of Mathematics, Warsaw Technical University, Warszawa, Poland

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In [1] BOSZNAY gives the following generalization of a theorem due to GELBAUM: Suppose $(K; d)$ is a compact connected metric space and μ is a probability measure on Borel subsets of K such that for $G \subset K$ open, non-empty $\mu(G) > 0$. Let $f, g \in C(K)$ be μ-stochastically independent, such that there exists $t_0 \in R$ with $0 < \text{card}\, f^{-1}(t_0) < \infty$. Then g is a constant function.

The purpose of this note is to give a very simple proof of the above theorem. Moreover we do not use a metric structure of K and we need weaker assumption on f, namely $0 < \text{card}\, f^{-1}(t_0) < \mathfrak{c}$ where $\mathfrak{c}$ is cardinality of the real line. We do not need compactness assumption in a very strong form, either. To make it clear we formulate the main result in abstract form.

Let us recall some standard terminology: Let (X, τ) be a topological space. We say that probability measure P defined on Borel subsets of X is regular if for every Borel set $A \subset X$

$$P(A) = \sup P(K)$$

where sup is taken over all compact sets $K \subset A$.

For regular measure P we define the support of P as the smallest closed set $A \subset X$ such that $P(A) = 1$. We denote it by supp (P).

The following lemmas are standard and we give proofs to make the paper self-contained.

LEMMA 1. Let X, Y be topological spaces, P be a probability regular measure on Borel σ-field in X and $f: X \to Y$ continuous. Then measure μ defined by $\mu(A) = P(f^{-1}(A))$ is regular. Moreover, $\text{supp}\, \mu = \overline{f(\text{supp}\, P)}$.

PROOF. Fix $\varepsilon > 0$ and $B \subset Y$ Borel. Then by regularity of P there exist a compact set $K \subset f^{-1}(A)$ such that $|P(f^{-1}(A)) - P(K)| < \varepsilon$. Clearly $\hat{K} = f(K)$ is compact, $\hat{K} \subset A$ and $|\mu(A) - \mu(\hat{K})| < \varepsilon$.

This proves regularity. Now let A be a closed set such, that $\mu(A) = 1$ i.e. $P(f^{-1}(A)) = 1$. Then, since $f^{-1}(A)$ is closed, $f^{-1}(A) \supset \text{supp}\,(P)$. Hence $f(\text{supp}\,(P)) \subset A$ and $\overline{f(\text{supp}\,(P))} \subset A$ because A is closed. Clearly

$$\mu\left(\overline{f(\overline{\operatorname{supp}}(P))}\right) = P\left(f^{-1}\left(\overline{f}(\overline{\operatorname{supp}}(P))\right)\right) \geq P(\operatorname{supp}(P)) = 1$$

therefore $\overline{f(\operatorname{supp}(P))}$ is the smallest closed set of measure 1.

LEMMA 2. If P, Q are regular probability measures on topological spaces X, Y respectively then

$$\operatorname{supp}(P\otimes Q) = \operatorname{supp}(P)\times \operatorname{supp}(Q).$$

PROOF. Clearly $\operatorname{supp}(P\otimes Q)\subset \operatorname{supp}(P)\times\operatorname{supp}(Q)$ always. To prove the converse inclusion let $x\in\operatorname{supp}(P)\times\operatorname{supp}(Q)-\operatorname{supp}(P\otimes Q)$. Since $\operatorname{supp}(P\otimes Q)$ is a closed set, one can choose open sets U, V, such that $x\in \in U\times V$ and $(U\times V)\cap\operatorname{supp}(P\otimes Q)=\emptyset$. Now since U (resp. V) is non-empty open set in $\operatorname{supp}(P)$ (resp. $\operatorname{supp}(Q)$), thus

$$(P\otimes Q)(U\times V) = P(U)\,Q(V)>0.$$

This contradicts to $\operatorname{supp}(P\otimes Q)\cap(U\times V)=\emptyset$.

COROLLARY 1. Let $f, g: X\to Y$ be continuous. Let P be a regular probability measure on X such that $X=\operatorname{supp}(P)$ and f, g are independent P-stochastically. Then $\overline{(f,g)(X)} = \overline{f(X)}\times\overline{g(X)}$.

PROOF. By lemma 1 measures μ, ν defined by formula

(1) $$\mu\otimes\nu(A) = P\left(f^{-1}(A)\cap g^{-1}(A)\right)$$

are regular. Thus $\overline{(f,g)(X)} = \operatorname{supp}\mu\otimes\nu = \operatorname{supp}\mu\times\operatorname{supp}\nu = \overline{f(X)}\times\overline{g(X)}$ by lemmas 1 and 2.

For every topological space X, which contains non-trivial connected subsets we define a cardinal number $\mathfrak{m}_X$ in the following way:

$$\mathfrak{m}_X = \min\{\mathfrak{n}:\mathfrak{n}=\operatorname{card}A,\ A\subset X \text{ closed connected, } \mathfrak{n}>1\}.$$

Clearly $\mathfrak{m}_R = \mathfrak{c}$. In next lemma we give lower bounds for $\mathfrak{m}_X$:

LEMMA 3. Suppose X contains connected subsets of cardinality >1.

(i) If X is Hausdorff space, then $\mathfrak{m}_X \geq \aleph_0$.
(ii) If X is complete metric space, then $\mathfrak{m}_X\geq\mathfrak{c}$.
(iii) If X is compact Hausdorff space, then $\mathfrak{m}_X\geq\mathfrak{c}$.

PROOF.

(i) Suppose that $\mathfrak{m}_X = n>1$. This means that there exists a closed connected set of the form $\{x_1,\ldots,x_n\}$. Let $U_i\ni x_1$ be open sets such that $x_i\notin U_i$ $i=2,3,\ldots,n$. Then $U=\bigcap_i U_i$ is an open set and $x_1\in U, x_2,\ldots,x_n\notin U$. This means that $\{x_1\}$ is an open set in relative topology generated from X. Since it is well-known that $\{x_1\}$ is closed set, we get a contradiction with the assumption that $\{x_1,\ldots,x_n\}$ is connected.

(ii) Let d be a metric on X such that $(X; d)$ is a complete space. Let $A\subset X$ be closed connected set with $\operatorname{card}(A)>1$. Then by (i) $\operatorname{card}(A)\geq\aleph_0$. We will construct $2^{\aleph_0}$ Cauchy sequences (z_n) with different limits in A.

Let $x_1 \neq y_1 \in A$ be arbitrary, and $z_1 \in \{x_1, y_1\}$. Suppose that $z_1, \ldots, z_{n-1}$ are defined as arbitrary elements $z_k \in \{x_k, y_k\}$ $k = 1, \ldots, n-1$. Since A is connected, it does not contain isolated points. Thus there exist $x_n \neq y_n$ such that $d(z_{n-1}; x_n) \leq \frac{1}{3} d(x_{n-1}; y_{n-1})$ and $d(z_{n-1}; y_n) \leq \frac{1}{3} d(x_{n-1}; y_{n-1})$. Let $z_n \in \{x_n, y_n\}$ be arbitrary. Then $d(z_n; z_{n+1}) \leq 3^{-n} d(x_1, y_1)$ thus (z_n) is Cauchy sequence. Let (z_n) and (z_n') be different sequences obtained by the above procedure (with the same points (x_n, y_n) if possible). Then $\lim z_n \neq \lim z_n'$, because, for some n, $d(z_n; z_n') = d(x_n, y_n) > 0$ and for every k $d(z_{n+k}; z_{n+k}') \geq \frac{1}{3} d(x_n, y_n)$. Since A is closed $\lim z_n \in A$, thus card $A \geq \mathfrak{c}$.

(iii) Idea of the proof is essentially the same as in (ii). It suffices to construct $2^{\aleph_0}$ decreasing sequences of compact sets $K_n \neq \emptyset$ disjoint for large n. To do this it suffices to use a standard property of compact Hausdorff sets:

(P): for every different points $x, y \in K$ there are open sets $U \ni x$, $V \ni y$ such that $\overline{U} \cap \overline{V} = \emptyset$.

Indeed, let $A \subset K$ be closed connected set with card $A > 1$. Let $x_1 \neq y_1$ be arbitrary elements of A. Since A is compact, using (P) we obtain U_1 and V_1. Let K_1 be $\overline{U}_1$ or $\overline{V}_1$. Since A does not contain isolated points, one can choose points $x_2 \neq y_2 \in K_1$ and once more apply (P) etc. By induction we obtain $2^{\aleph_0}$ decreasing sequences (K_n). However A is compact, thus $\bigcap_n K_n \neq \emptyset$ and card $A > \mathfrak{c}$.

The main result of this note is the following proposition:

PROPOSITION. *Let X, Y be topological spaces, P be a regular probability measure on Borel subsets of X. Assume that $f, g: X \to Y$ are non-constant continuous P — independent functions such that*

$(f, g)(\text{supp } P)$ *is closed,*
$g(\text{supp } P)$ *is connected.*

Then for every $y \in f(\text{supp}(P))$ card $f^{-1}(y) \geq \mathfrak{m}_Y$.

PROOF. By corollary 1 $(f, g)(\text{supp}(P)) = f(\text{supp } P) \times g(\text{supp } P)$. Thus if $y \in \text{supp } P$ then card $f^{-1}(y) \geq$ card $(\{y\} \times g(\text{supp } P)) \geq \mathfrak{m}_Y$.

COROLLARY 2. If P is a regular probability measure on a connected nontrivial topological space X and f, g are continuous P — independent mappings: $X \to Y$ such that $(f, g)(X)$ is closed and for some $y_0 \in Y$

$$0 < \text{card } f^{-1}(y_0) < \min\{\mathfrak{m}_X; \mathfrak{m}_Y\} \tag{2}$$

then condition

$$\text{for every open non-empty } G \subset X, \quad P(G) > 0 \tag{3}$$

implies that g is a constant function.

PROOF. First let us mention that condition (3) means $X = \text{supp } P$. Second observation is, that $f \neq$ const. Indeed, suppose f is constant. Then

$X = f^{-1}(y_0)$, thus by (2) card $X < \mathfrak{m}_X$ which is impossible. Since $g(X)$ is a connected set, the proof is finished by Proposition.

Now we will state in explicit form a generalization of BOSZNAY's result.

COROLLARY 3. Let P be a probability measure on a compact Hausdorff connected set K such that for every open non-empty $G \subset K$ $P(G) > 0$. Assume that for some topological Hausdorff space Y $f, g: K \to Y$ are continuous P-independent functions such that for some $y_0 \in Y$ $0 < \text{card } f^{-1}(y_0) < \mathfrak{c}$.

Then g is a constant function.

PROOF. It sufficies to apply corollary 2. Since K is compact, so is $(f, g)(K)$. Thus by lemma 3 assumptions of corollary 2 are satisfied if K is not one-point set. Clearly if card $(K) = 1$, then $g = \text{const}$ either.

ADDED IN PROOF: The author wishes to thank G. ZBAGANU for pointing out an error in the formulation of Corollary 1.

In the case $X = \langle 0, 1 \rangle$ Corollary 1 was proved by J. HOLBROOK: Stochastic independence and space-filling curves, *Am. Math. Monthly*, **88** (1981), 426–432.

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THE MAXIMAL ELEMENTS OF A PARTIALLY ORDERED SET AND A MODIFICATION OF THE THEOREM OF KREIN—MILMAN FOR CLOSURE OPERATORS

By

R. THRON

Pädagogische Hochschule N. K. Krupskaja, Halle, DDR

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In the present paper the set of the maximal elements of some subset X of a partially ordered set L is described by the set $[X]_T$ of the T-isolated elements of X where T is a closure operator on L. For that purpose a modification of the theorem of KREIN—MILMAN is used. By this description the set of the maximal elements of X is equal to the set of the maximal elements of $[X]_T$. Analogously, there exists a description of the set of the minimal elements of X, too.

0. Introduction and results

By the theorem of KREIN—MILMAN every compact and convex subset X of a locally convex space is equal to the closed convex closure of the set of the extremal points of X (see KREIN—MILMAN [4], KÖTHE [5]). In this paper there is concluded a modification of the theorem of KREIN—MILMAN for closure operators. For that purpose let L be a set. The power set of L is denoted by $P(L)$.

Then a map T from $P(L)$ into $P(L)$ is called a *closure operator* on L iff the following conditions hold for $X, Y \in P(L)$ (see JÓNSSON [3]).

(0) $X \subseteq T(X)$.
(1) From $X \subseteq Y$ it follows that $T(X) \subseteq T(Y)$.
(2) $T(T(X)) = T(X)$.

Any element x of X is called *T-isolated* iff $x \notin T(X - \{x\})$ (see SCHMIDT [6]). The set of all T-isolated elements of X is denoted by $[X]_T$.

There is given a sufficient condition such that $X = T([X]_T)$ holds for some subset X of L.

Now let L be partially ordered by the irreflexive relation $<$. Let max (Z) be the set of all *maximal* elements of Z for $Z \subseteq L$. For any element z of Z one has $z \in \max(Z)$ iff there is no element z_0 of Z such that $z < z_0$ (see GRÄTZER [2]). Then in $P(L)$ there exists an *equivalence relation* γ such that $\gamma = \{(X, Y): X, Y \in P(L).\ \max(X) = \max(Y).\}$.

Let $\mathfrak{M}$ be a family of sets. Then the union of all elements of $\mathfrak{M}$ is denoted by $\bigcup\mathfrak{M}$. If $\mathfrak{M}\neq\emptyset$, then the intersection of all elements of $\mathfrak{M}$ is denoted by $\bigcap\mathfrak{M}$.

THEOREM 1. *Let $\gamma_0\subseteq\gamma$. Assume that from*

$$\mathfrak{T}\subseteq\gamma_0\cup\{(X, Y):(Y, X)\in\gamma_0.\}\cup\{(X, X):X\in P(L).\}$$

it follows that

$$\left(\bigcup\{X:(X, Y)\in\mathfrak{T}.\},\quad \bigcup\{Y:(X, Y)\in\mathfrak{T}.\}\right)\in\gamma.$$

Then the following holds.

(0) *A closure operator T on L is defined by the formula*

$$T(X)=\bigcap\{Y:X\subseteq Y\in P(L).$$
$$\textit{From } (U, W)\in\gamma_0 \textit{ it follows that } U\subseteq Y \textit{ iff } W\subseteq Y.\}$$

for $X\in P(L)$.

(1) *If $X_0\in P(L)$ and $X_0\subseteq X\subseteq T(X_0)$, then $(X_0, X)\in\gamma$.* ▮

There exists a subset γ_0 of γ, satisfying the assumption of theorem 1, by

THEOREM 2. *Let $\gamma_0\subseteq\gamma$. Assume that for every element (U, W) of γ_0 the sets U, W are finite. Then from*

$$\mathfrak{T}\subseteq\gamma_0\cup\{(X, Y):(Y, X)\in\gamma_0.\}\cup\{(X, X):X\in P(L).\}$$

it follows that

$$\left(\bigcup\{X:(X, Y)\in\mathfrak{T}.\},\quad \bigcup\{Y:(X, Y)\in\mathfrak{T}.\}\right)\in\gamma. \text{ ▮}$$

Now let γ_0 be a subset of γ, satisfying the assumption of theorem 1. Furthermore, let T be the closure operator on L, existing by theorem 1. Obviously, then there exists a subset $\mathfrak{L}$ of $L\times P(L)$ such that

$$T(X)=\bigcap\{Y:X\subseteq Y\subseteq L.$$
$$\text{From } (x_0, X_0)\in\mathfrak{L} \text{ and } X_0\subseteq Y \text{ it follows that } x_0\in Y.\}.$$

Assume that there exists a ternary relation $\mathfrak{F}$ in $P(L)$, i.e. $\mathfrak{F}\subseteq P(L)\times\times P(L)\times P(L)$, such that the following conditions hold where N denotes the set of the natural numbers.

(3) From $(F^{-1}, F, F^1)\in\mathfrak{F}$ it follows that $F^{-1}\cap F^1\subseteq F$.
(4) There exists no subset $\{(F_i^{-1}, F_i, F_i^1):i\in N.\}$ of $\mathfrak{F}$ such that

$$(F_0\cap F_1\cap\cdots\cap F_i)\supset(F_0\cap F_1\cdots\cap F_i\cap F_{i+1})$$

for every natural number i.

(5) If $(x, X)\in\mathfrak{L}$, $(F^{-1}, F, F^1)\in\mathfrak{F}$ and $x\in F^{-1}\cap F^1$, then one of the following conditions holds.

(5.1) There exists an element x_{-1} of X such that $x_{-1}\notin F^1$.
(5.2) $X\subseteq F^{-1}\cap F^1$.

Let $X \subseteq L$. Then the $\mathfrak{F}$-*closure* $\mathfrak{F}(X)$ of X is defined by the formula

$$\mathfrak{F}(X) = \bigcap\big(\{L\} \cup \{F^1 : X \subseteq F^1. \text{ There exists an element } (F^{-1}, F, F^1) \text{ of } P(L)\times P(L)\times P(L) \text{ such that } (F^{-1}, F, F^1)\in\mathfrak{F}.\}\big).$$

$\mathfrak{F}(X)$ is called $(\mathfrak{Q}, \mathfrak{F})$-*compact* iff from $(x_0, X_0)\in\mathfrak{Q}$, $(x_0, \emptyset)\notin\mathfrak{Q}$, $x_0\in\mathfrak{F}(X)$ it follows that there exists an element $(x_0, \tilde{X}_0)$ of $\mathfrak{Q}$ with the properties. For $x\in\tilde{X}_0$ there exists an element (F_x^{-1}, F_x, F_x^1) of $\mathfrak{F}$ such that

$$(X_0 \cup \{x_0\} \cup \tilde{X}_0) \cap F_x = \{x\} \subseteq \mathfrak{F}(X) \cap F_x^{-1}$$

and $\mathfrak{F}(X) \subseteq F_x^1$.

Now one has the announced modification of the theorem of KREIN—MILMAN.

THEOREM 3. *Let* $X \subseteq L$. *If* $\mathfrak{F}(X)$ *is* $(\mathfrak{Q}, \mathfrak{F})$-*compact and* $\mathfrak{F}(X) = T(\mathfrak{F}(X))$, *then* $\mathfrak{F}(X) = T([\mathfrak{F}(X)]_T)$. ▌

One obtains the theorem of Krein—Milman for an Euclidean vector space L if the following three assumptions hold.

(6) If $X \subseteq L$, then $T(X)$ is the convex closure of X.

(7) $\mathfrak{Q} = \{(x, X) : (x, X)\in L\times P(L)$. There exist collinear points x_0, x_1, x_2 of L such that $x = x_0$, $X = \{x_1, x_2\}$ and x_0 is between x_1 and x_2.$\}$.

(8) $(F^{-1}, F, F^1)\in\mathfrak{F}$ iff F is a hyperplane of L, F^{-1} and F^1 are closed halfspaces generated by F with $F^{-1} \cap F^1 = F$.

Let X be a bounded subset of L. Then $\mathfrak{F}(X)$ is $(\mathfrak{Q}, \mathfrak{F})$-compact and $\mathfrak{F}(X) = T(\mathfrak{F}(X))$. $[\mathfrak{F}(X)]_T$ is the set of the extremal points of $\mathfrak{F}(X)$.

Finally, from theorem 1 and theorem 3 one obtains the following theorems.

THEOREM 4. *Let* $X \subseteq L$. *If* $\mathfrak{F}(X)$ *is* $(\mathfrak{Q}, \mathfrak{F})$-*compact and* $\mathfrak{F}(X) = T(\mathfrak{F}(X))$, *then* $(\mathfrak{F}(X), [\mathfrak{F}(X)]_T)\in\gamma$. ▌

THEOREM 5. *Let* $X \subseteq L$. *If* $\mathfrak{F}(X)$ *is* $(\mathfrak{Q}, \mathfrak{F})$-*compact and* $\mathfrak{F}(X) = T(X)$, *then* $(X, [X]_T)\in\gamma$, $(X, \mathfrak{F}(X))\in\gamma$, $(\mathfrak{F}(X), [\mathfrak{F}(X)]_T)\in\gamma$. ▌

1. The proofs of the theorems

LEMMA 1. Let h be a map from $P(L)$ into $P(L)$ with the following properties for $X, Y\in P(L)$.

(0) $X \subseteq h(X)$.

(1) $h(X) \subseteq h(Y)$ for $X \subseteq Y$.

Then there exists a closure operator T on L such that

$$T(X_0) = \bigcap\{X : X_0 \subseteq X\in P(L).\ h(X) = X.\} \quad \text{for} \quad X_0\in P(L).$$

From

$$\mathfrak{H}(X_0) = \bigcap\{\mathscr{H} : X_0\in\mathscr{H} \subseteq P(L).\ \text{If } X\in\mathscr{H}, \text{ then } h(X)\in\mathscr{H}.\ \text{If } \mathfrak{Y} \subseteq \mathscr{H}, \text{ then } \bigcup\mathfrak{Y}\in\mathscr{H}.\}$$

it follows that $\mathfrak{H}(X_0)$ is well-ordered by set inclusion (see BACHMANN [1]) and the following holds.

(2) $\mathrm{T}(X_0)$ is the greatest element of $\mathfrak{H}(X_0)$.

(3) X_0 is the least element of $\mathfrak{H}(X_0)$.

(4) The successor of any element X of $\mathfrak{H}(X_0)$ is $h(X)$.

(5) For every limit-element X of $\mathfrak{H}(X_0)$ different from X_0 one has $X = \bigcup\{Y : Y \subset X,\ Y \in \mathfrak{H}(X_0).\}$.

PROOF. Let $\mathcal{M}$ be the family of all subsets $\mathfrak{A}$ of $\mathfrak{H}(X_0)$ well-ordered by setinclusion with the following properties. X_0 is the least element of $\mathfrak{A}$.

The successor of any element X of $\mathfrak{A}$ is $h(X)$.

For every limit-element X of $\mathfrak{A}$ different from X_0 one has

$$X = \bigcup\{Y : Y \subset X,\ Y \in \mathfrak{A}.\}.$$

From $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}$ it follows that $\mathfrak{A}$ is a section of $\mathfrak{G}$ or $\mathfrak{G}$ is a section of $\mathfrak{A}$ or $\mathfrak{A} = \mathfrak{G}$. Without loss of generality assume that $\mathfrak{A} - \mathfrak{G} \neq \emptyset$. Let A be the least element of $\mathfrak{A} - \mathfrak{G}$ and assume that $R(A) = \{X : X \in \mathfrak{A}.\ X \subset A.\}$. It follows that $R(A) \subseteq \mathfrak{G}$. Moreover $R(A) = \mathfrak{G}$.

The assumption $R(A) \subset \mathfrak{G}$ leads to a contradiction as follows. Let B be the least element of $\mathfrak{G} - R(A)$. Since $X_0 \in \mathfrak{G} \cap R(A)$, one has $X_0 \subset B$.

Then two cases are possible. $B = h(B')$ and $B' \in R(A)$ hold. Hence $B' \in \mathfrak{A}$ and $B' \subset A$. It follows that $B \in \mathfrak{A}$ and $B \subseteq A$. Assume that $B = \bigcup\{B' : B' \subset B.\ B' \in \mathfrak{G}.\}$ and $B' \in R(A)$. Hence $B' \in \mathfrak{A}$ and $B' \subset A$. Thus $B \in \mathfrak{A}$ and $B \subseteq A$, too. Since $A \notin \mathfrak{G}$, one obtains that $B = A$ is impossible. Therefore $B \subset A$ $B \in R(A)$, a contradiction.

Assume that $\mathfrak{B} = \bigcup\{\mathfrak{A} : \mathfrak{A} \in \mathcal{M}.\}$. Then $\mathfrak{B} \in \mathcal{M}$. First $\mathcal{M}$ is ordered by setinclusion. Then $\mathfrak{B}$ is ordered. From $X, Y \in \mathfrak{B}$ and $\mathfrak{A}, \mathfrak{G} \in \mathcal{M}$ and $X \in \mathfrak{A}$, $Y \in \mathfrak{G}$ it follows that $X, Y \in \mathfrak{A}$ without loss of generality, Hence $X = Y$ or $X \subset Y$ or $Y \subset X$. Moreover $\mathfrak{B}$ is well-ordered. Assume that $\emptyset \neq \mathfrak{W} \subseteq \mathfrak{B}$. Then there exists an element $\mathfrak{A}$ of $\mathcal{M}$ such that $\mathfrak{W} \cap \mathfrak{A} \neq \emptyset$. From $\emptyset \neq \mathfrak{W} \cap \mathfrak{A}$ it follows that there exists the least element W_0 of $\mathfrak{W} \cap \mathfrak{A}$. W_0 is the least element of $\mathfrak{W}$, too. Otherwise there exists an element W_1 of $\mathfrak{W}$ such that $W_1 \subset W_0$. Since W_0 is the least element of $\mathfrak{W} \cap \mathfrak{A}$, one has $W_1 \notin \mathfrak{A}$.

Therefore there exists an element $\mathfrak{G}$ of $\mathcal{M}$ such that $\mathfrak{A} \subset \mathfrak{G}$ and $W_1 \in \mathfrak{G}$. Moreover $\mathfrak{A} = \{X : X \in \mathfrak{G}.\ X \subset B.\}$ where $B \in \mathfrak{G}$. Therefore $W_0 \subset B$ and $B \subseteq W_1$. Hence $W_0 \subset W_1$, a contradiction. Assume that the element X^* of $\mathfrak{B}$ is the successor of an element X of $\mathfrak{B}$. Then there exists an element $\mathfrak{A}$ of $\mathcal{M}$ such that $X, X^* \in \mathfrak{A}$ and $h(X) = X^*$. Finally, let X be a limit-element of $\mathfrak{B}$. Assume that $X \neq X_0$. Then there exists an element $\mathfrak{A}$ of $\mathcal{M}$ such that $X \in \mathfrak{A}$ and X is a limit-element of $\mathfrak{A}$. Therefore

$$X = \bigcup\{Y : Y \subset X.\ Y \in \mathfrak{A}.\} = \bigcup\{Y : Y \subset X.\ Y \in \mathfrak{B}.\}.$$

Now it follows that $\mathfrak{B} = \mathfrak{H}(X_0)$. First $\mathfrak{B}$ is the greatest element of $\mathcal{M}$ in regard to setinclusion. Moreover, one has $\mathfrak{B} \subseteq \mathfrak{H}(X_0)$. The assumption of $\mathfrak{B} \subset \mathfrak{H}(X_0)$ leads to a contradiction as follows. Two cases are possible. There exists an element A' of $\mathfrak{B}$ such that $A = h(A') \notin \mathfrak{B}$. Then A' is the greatest element of $\mathfrak{B}$. Therefore $\mathfrak{B} \subset \mathfrak{B} \cup \{A\}$ such that $\mathfrak{B} \subset \{A\} \in \mathcal{M}$, a contradiction.

There exists a subset $\mathscr{H}$ of $\mathfrak{V}$ such that $\bigcup\mathscr{H}\notin\mathfrak{V}$. Then there does not exist an element A of $\mathfrak{V}$ such that $X\subset A$ where $X\in\mathscr{H}$. Otherwise let A^* be the least element of $\mathfrak{V}$ such that $X\subset A^*$ where $X\in\mathscr{H}$. A^* is not limit-element. Otherwise $\bigcup\mathscr{H}=A^*$, contradicting $\bigcup\mathscr{H}\notin\mathfrak{V}$. Therefore A^* is isolated. Now there exists an element A' of $\mathfrak{V}$ such that $h(A')=A^*$ and $A'\subset A^*$. From the assumption on A^* it follows that A' is the greatest element of $\mathscr{H}$ and $A'=\bigcup\mathscr{H}$, contradicting $\bigcup\mathscr{H}\notin\mathfrak{V}$. Therefore $\bigcup\mathscr{H}=\bigcup\mathfrak{V}\notin\mathfrak{V}$ and $B\subset\bigcup\mathfrak{V}$ where $B\in\mathfrak{V}$. One has $\mathfrak{V}\subset\mathfrak{V}\cup\{\bigcup\mathfrak{V}\}$ and $\mathfrak{V}\cup\{\bigcup\mathfrak{V}\}\in\mathscr{M}$ contrary to the fact that $\mathfrak{V}$ is the greatest element of $\mathscr{M}$.

$T(X_0)$ is the greatest element of $\mathfrak{H}(X_0)$. First $\bigcup\mathfrak{H}(X_0)$ is the greatest element of $\mathfrak{H}(X_0)$. It follows that

$$\bigcup\mathfrak{H}(X_0)\subseteq h\big(\bigcup\mathfrak{H}(X_0)\big)\subseteq\bigcup\mathfrak{H}(X_0).$$

Hence $\bigcup\mathfrak{H}(X_0)=h\big(\bigcup\mathfrak{H}(X_0)\big)$. Since $X_0\subseteq\bigcup\mathfrak{H}(X_0)$, one has $T(X_0)\subseteq\bigcup\mathfrak{H}(X_0)$. Moreover $T(X_0)=\bigcup\mathfrak{H}(X_0)$. Otherwise $T(X_0)\subset\bigcup\mathfrak{H}(X_0)$. Then there exists an element X of $\mathfrak{H}(X_0)$ such that $X\nsubseteq T(X_0)$. Now let X^* be the least element of $\mathfrak{H}(X_0)$ such that $X^*\nsubseteq T(X_0)$. Then two cases are possible. X^* is isolated. Then there exists an element X' such that $X'\subset X^*$, $h(X')=X^*$ and $X'\subseteq T(X_0)$.

It follows that $X^*\subseteq h(T(X_0))=T(X_0)$, a contradiction. X^* is a limit-element. Then $X^*=\bigcup\{X:X\subset X^*,\ X\in\mathfrak{H}(X_0).\}$ and from $X\in\mathfrak{H}(X_0)$ and $X\subset X^*$ it follows that $X\subseteq T(X_0)$. Then $X^*\subseteq T(X_0)$, a contradiction, too.

Finally, a closure operator T on L is defined by the formula

$$T(X_0)=\bigcap\{X:X_0\subseteq X\in P(L).\ h(X)=X.\}$$

for $X_0\in P(L)$. Obviously, from $X\subseteq Y\in P(L)$ it follows that $X\subseteq T(X)$ and $T(X)\subseteq T(Y)$. Since $h(T(X))=T(X)$ it follows that $T(T(X))=T(X)$. ∎

Proof of theorem 1. Let h be a map from $P(L)$ into $P(L)$ such that

$$h(X)=X\cup\bigcup\{Y:(Y,Z)\in\gamma_0,\ Z\subseteq X.\}\cup\bigcup\{Z:(Y,Z)\in\gamma_0,\ Y\subseteq X.\}$$

where $X\in P(L)$.

If $X\subseteq Y\in P(L)$, then obviously $X\subseteq h(X)$ and $h(X)\subseteq h(Y)$. Now $h(X)=X$ iff the following holds. From $(Y,Z)\in\gamma_0$ it follows that $Y\subseteq X$ iff $Z\subseteq X$.

From lemma 1 one obtains that a closure operator T on L is defined by the formula

$$T(X)=\bigcap\{Y:X\subseteq Y\in P(L).$$
$$\text{From } (U,W)\in\gamma_0 \text{ it follows that } U\subseteq Y \text{ iff } W\subseteq Y.\}$$

for $X\in P(L)$.

Let $X_0\in P(L)$. By lemma 1 one has a set $\mathfrak{H}(X_0)$ such that $X_0,\ T(X_0)\in\mathfrak{H}(X_0)$. Now from $X\in\mathfrak{H}(X_0)$ one concluds that $(X_0,X)\in\gamma$. First one has $(X_0,X_0)\in\gamma$. Assume that there exists an element X of $\mathfrak{H}(X_0)$ such that $(X_0,X)\notin\gamma$. Then in relation to the well-ordering of $\mathfrak{H}(X_0)$ there exists a least element X^* of $\mathfrak{H}(X_0)$ such that $(X_0,X^*)\notin\gamma$ and $X_0\subset X^*$. Two cases are possible. X^* is isolated. Hence there exists an element X' of $\mathfrak{H}(X_0)$ such that $h(X')=X^*$ and $(X_0,X')\in\gamma$. It follows that $(X',h(X'))\in\gamma$. Assume that

$\mathfrak{T} = \{(Y, Z) : (Y, Z) \in \gamma_0.\ Z \subseteqq X'.\} \cup \{(Y, Z) : (Z, Y) \in \gamma_0.\ Z \subseteqq X'.\} \cup \{(X', X')\}$. Then from the assumption on γ_0 it follows that

$$\left(\bigcup\{Y : (Y, Z) \in \mathfrak{T}.\}, \bigcup\{Z : (Y, Z) \in \mathfrak{T}.\}\right) = \left(h(X'), X'\right) \in \gamma .$$

Therefore $(X_0, X^*) \in \gamma$, a contradiction.

X^* is a limit-element. Then $X^* = \bigcup\{Y : Y \subset X^*.\ Y \in \mathfrak{H}(X_0).\}$. From $Y \in \mathfrak{H}(X_0)$ and $Y \subset X^*$ one obtains that $(X_0, Y) \in \gamma$. One has $\max\left(\bigcup\{Y : Y \subset X^*.\ Y \in \mathfrak{H}(X_0).\}\right) \subseteqq \max(X_0)$. Assume that $x_0 \in \max\left(\bigcup\{Y : Y \subset X^*.\ Y \in \mathfrak{H}(X_0).\}\right)$. Then there exists an element Y of $\mathfrak{H}(X_0)$ such that $Y \subset X^*$ and $x_0 \in \max(Y) = \max(X_0)$. Furthermore, $\max(X_0) \subseteqq \max\left(\bigcup\{Y : Y \subset X^*.\ Y \in \mathfrak{H}(X_0).\}\right)$. Assume that $x_0 \in \max(X_0)$. The assumption of $x_0 \notin \max\left(\bigcup\{Y : Y \subset X^*.\ Y \in \mathfrak{H}(X_0).\}\right)$ leads to a contradiction as follows. Then there exists an element Y of $\mathfrak{H}(X_0)$ such that $Y \subset X^*$ and $x_0 \notin \max(Y) = \max(X_0)$. Therefore $(X_0, X^*) \in \gamma$, contradicting the assumption on X^*.

Now it follows that $\left(X_0, T(X_0)\right) \in \gamma$. From $X_0 \subseteqq X \subseteqq T(X_0)$ one obtains that $T(X) = T(X_0)$ and therefore $\left(X_0, T(X)\right) \in \gamma$. Furthermore $\left(X, T(X)\right) \in \gamma$. From this fact and $\left(X_0, T(X)\right) \in \gamma$ it follows that $(X_0, X) \in \gamma$. ∎

PROOF OF THEOREM 2. Assume that $\mathfrak{Z} \subseteqq P(L)$ such that every element Z of $\mathfrak{Z}$ is finite. Then

$$\max\left(\bigcup \mathfrak{Z}\right) = \max\left(\bigcup\{\max(Z) : Z \in G.\}\right).$$

From $z_0 \in \max(\bigcup \mathfrak{Z})$ one obtains that there exists an element Z_0 of $\mathfrak{Z}$ such that $z_0 \in Z_0$. Then $z_0 \in \max(Z_0)$ and $z_0 \in \max\left(\bigcup\{\max(Z) : Z \in \mathfrak{Y}.\}\right)$.

Let z_0 be an element of $\max\left(\bigcup\{\max(Z) : Z \in \mathfrak{Z}.\}\right)$ such that $z_0 \in \max(Z_0) \subseteqq Z_0 \in \mathfrak{Z}$. Assume that there exists an element z_1 of $\max(\bigcup \mathfrak{Z})$ such that $z_0 < z_1 \in Z_1 \in \mathfrak{Z}$. Since Z_1 is finite, there exists an element z_1^* such that $z_0 < z_1 \leqq z_1^* \in \max(Z_1)$, a contradiction.

Since

$$(X, X) = \left(\bigcup\{\{x\} : x \in X.\}, \bigcup\{\{x\} : x \in X.\}\right)$$

there is assumed without loss of generality that for every element (U, W) of $\mathfrak{T}$ the sets U, W are finite.

Finally, it follows that

$$\max\left(\bigcup\{X : (X, Y) \in \mathfrak{T}.\}\right) = \max\left(\bigcup\{\max(X) : (X, Y) \in \mathfrak{T}.\}\right) =$$
$$= \max\left(\bigcup\{\max(Y) : (X, Y) \in \mathfrak{T}.\}\right) = \max\left(\bigcup\{Y : (X, Y) \in \mathfrak{T}.\}\right). ∎$$

LEMMA 2. Let h be a map from $P(L)$ into $P(L)$ such that the following holds for $X, Y \in P(L)$.

(0) $X \subseteqq h(X)$.

(1) From $X \subseteqq Y$ it follows that $h(X) \subseteqq h(Y)$.

Furthermore, let $[X]_h$ be defined by the formula

$$[X]_h = \{x : x \in X.\ x \notin h(X - \{x\}).\}.$$

If T is the closure operator on L, existing by lemma 1, then $[T(X)]_T = [T(X)]_h$.

PROOF. From $x \in T(X)$ it follows that $x \in T(T(X)-\{x\})$ iff $x \in h(T(X)-\{x\})$. Obviously, from $x \in h(T(X)-\{x\})$ one obtains that $x \in T(T(X)-\{x\})$.

Conversely, let $x \in T(T(X)-\{x\})$. Consider the set $\mathfrak{H}(T(X)-\{x\})$, existing by lemma 1. Then $T(X)$ is the greatest element of $\mathfrak{H}(T(X)-\{x\})$. From this fact it follows that $x \in h(T(X)-\{x\})$. Otherwise, $T(X)-\{x\}$ is the greatest element of $\mathfrak{H}(T(X)-\{x\})$, contradicting $(T(X)-\{x\}) \subset T(X)$. ∎

LEMMA 3. There exists no subset $\{\tilde{\mathfrak{F}}_i : i \in N.\}$ of $P(\tilde{\mathfrak{F}})$ such that the following conditions hold.

(0) $\tilde{\mathfrak{F}}_i \subset \tilde{\mathfrak{F}}_{i+1}$ for $i \in N$.

(1) $\bigcap\{F$: There exists an element (F^{-1}, F, F^1) of $P(L)\times P(L)\times P(L)$ such that $(F^{-1}, F, F^1) \in \tilde{\mathfrak{F}}_i.\} \supset$

$\supset \bigcap\{F$: There exists an element (F^{-1}, F, F^1) of $P(L)\times P(L)\times P(L)$ such that $(F^{-1}, F, F^1) \in \tilde{\mathfrak{F}}_{i+1}.\}$ for every natural number i.

PROOF. Assume that the lemma does not hold. Then there exists for $i \in N$ an element $(F_{i+1}^{-1}, F_{i+1}, F_{i+1}^1)$ of $\tilde{\mathfrak{F}}_{i+1}$ such that

$$\bigcap\{F: \text{There exists an element } (F^{-1}, F, F^1) \text{ of } P(L)\times P(L)\times P(L) \text{ such that } (F^{-1}, F, F^1) \in \tilde{\mathfrak{F}}_i.\} \supset$$

$$\supset \bigcap\{F: \text{There exists an element } (F^{-1}, F, F^1) \text{ of } P(L)\times P(L)\times P(L) \text{ such that } (F^{-1}, F, F^1) \in \tilde{\mathfrak{F}}_i.\} \cap F_{i+1}.\}$$

Let $(F_0^{-1}, F_0, F_0^1) \in \tilde{\mathfrak{F}}_0$. Then

$$(F_0 \cap F_1 \cap \ldots \cap F_i) \supset (F_0 \cap F_1 \cap \ldots \cap F_i \cap F_{i+1})$$

for every natural number i, a contradiction. ∎

Let q be a map from a set A into a set B.

Furthermore, let X be a subset of A. Then the image $q\langle X\rangle$ of X is defined by the formula $q\langle X\rangle = \{q(x) : x \in X.\}$.

LEMMA 4. Let A and B be sets. Moreover, let B be partially ordered by the irreflexive relation R. There is assumed that there exists no subset $\{y_i : i \in N.\}$ of B such that

$$(y_1, y_0) \in R, \quad (y_2, y_1) \in R, \ldots, (y_{i+1}, y_i) \in R$$

for every natural number i.

Let q be a map from A into B. Finally, let S be a propositional function with the domain A such that the following condition holds.

(0) Let $x \in A$. If $S(z)$ is true for every element z of A with $(q(z), q(x)) \in R$, then $S(x)$ is true.

Then $S(x)$ is true for $x \in A$.

PROOF. Assume that there exists an element x_0 of A such that $S(x_0)$ is false. Then there exist a natural number n and elements $q(x_1), q(x_2), \ldots, q(x_n)$ of $q\langle A\rangle$ such that

$$(q(x_1), q(x_0)) \in R, \quad (q(x_2), q(x_1)) \in R, \ldots, (q(x_n), q(x_{n-1})) \in R$$

where $S(x_0), S(x_1), \ldots, S(x_n)$ are false and $S(z)$ is true for every element z of A with $(\varphi(z), \varphi(x_n))\in R$. Then $S(x_n)$ is true, a contradiction. ∎

PROOF OF THEOREM 3. Let h be a map from $P(L)$ into $P(L)$ such that $h(X) = X\cup\{x_0$: There exists an element (x_0, X_0) of $\mathfrak{Q}$ such that $X_0\subseteqq X.\}$.

Then $X\subseteqq h(X)$, $h(X)\subseteqq h(Y)$ for $X\subseteqq Y\in P(L)$.

Then from lemma 1, lemma 2 and $\mathfrak{F}(X)\subseteqq T([\mathfrak{F}(X)]_h)$ it follows that theorem 3 is true. Therefore it is enough to show that $\mathfrak{F}(X)\subseteqq T([\mathfrak{F}(X)]_h)$.

First, $x_0\in[\mathfrak{F}(X)]_h$ iff $x_0\in\mathfrak{F}(X)$ and from $(x_0, X_0)\in\mathfrak{S}$, $X_0\subseteqq\mathfrak{F}(X)$ it follows that $x_0\in X_0$.

For $x_0\in\mathfrak{F}(X)$ one has $x_0\in T([\mathfrak{F}(X)]_h)$.

Obviously, from $x_0\in\mathfrak{F}(X)$ and $(x_0, \emptyset)\in\mathfrak{Q}$ it follows that $x_0\in T([\mathfrak{F}(X)]_h)$.

Let $[\mathfrak{F}(X)]_0$ be the set of all elements x_0 of $\mathfrak{F}(X)$ such that $(x_0, \emptyset)\notin\mathfrak{Q}$ and there exists an element (F^{-1}, F, F^1) of $\mathfrak{F}$ with $x_0\in F^{-1}\cap F^1$ and $\mathfrak{F}(X)\subseteqq$ $\subseteqq F^1$.

Let $P(\mathfrak{F})$ be partially ordered by the irreflexive relation R in such a way that from $\mathfrak{F}_1, \mathfrak{F}_2\in P(\mathfrak{F})$ one obtains $(\mathfrak{F}_1, \mathfrak{F}_2)\in R$ iff following conditions hold.

(0) $\mathfrak{F}_2\subset\mathfrak{F}_1$.

(1) $\bigcap\{F$: There exists an element (F^{-1}, F, F^1) of $P(L)\times P(L)\times P(L)$ such that $(F^{-1}, F, F^1)\in\mathfrak{F}_1.\}\subset$
$\subset\bigcap\{F$: There exists an element (F^{-1}, F, F^1) of $P(L)\times P(L)\times P(L)$ such that $(F^{-1}, F, F^1)\in\mathfrak{F}_2.\}$.

It follows from lemma 3 that there exists no subset $\{y_i : i\in N.\}$ of $P(\mathfrak{F})$ such that

$$(y_1, y_0)\in R, \quad (y_2, y_1)\in R, \ldots, (y_{i+1}, y_i)\in R$$

for every natural number i.

Furthermore, let φ be a function from $[\mathfrak{F}(X)]_0$ into $P(\mathfrak{F})$ such that from $x_0\in[\mathfrak{F}(X)]_0$ it follows that

$$\varphi(x_0) = \{(F^{-1}, F, F^1) : (F^{-1}, F, F^1)\in\mathfrak{F}.\ x_0\in F^{-1}\cap F^1.\ \mathfrak{F}(X)\subseteqq F^1.\}.$$

Now there exists for $x_0\in([\mathfrak{F}(X)]_0-[\mathfrak{F}(X)]_h)$ an element $(x_0, \tilde{X}_0)$ of $\mathfrak{Q}$ such that $\tilde{X}_0\subseteqq[\mathfrak{F}(X)]_0$ and $(\varphi(x), \varphi(x_0))\in R$ for $x\in\tilde{X}_0$.

First, from $(x_0, X_0)\in\mathfrak{Q}$, $(F^{-1}, F, F^1)\in\mathfrak{F}$, $x_0\in F^{-1}\cap F^1$, $\mathfrak{F}(X)\subseteqq F^1$ and $X_0\subseteqq\mathfrak{F}(X)$ one obtains $X_0\subseteqq F^{-1}\cap F^1$. Otherwise, there exists an element x_{-1} of X_0 such that $x_{-1}\notin F^1$. Therefore $x_{-1}\notin\mathfrak{F}(X)$, a contradiction.

Now let x_0 be an element of $([\mathfrak{F}(X)]_0-[\mathfrak{F}(X)]_h)$. Then there exists an element (x_0, X_0) of $\mathfrak{G}$ such that $X_0\subseteqq\mathfrak{F}(X)$ and $x_0\notin X_0$. Since $\mathfrak{F}(X)$ is $(\mathfrak{G}, \mathfrak{F})$-compact there exists an element $(x_0, \tilde{X}_0)$ of $\mathfrak{Q}$ such that the following holds. For every element x of $\tilde{X}_0$ there exists an element (F_x^{-1}, F_x, F_x^1) of $\mathfrak{F}$ with

$$(X_0\cup\{x_0\}\cup\tilde{X}_0.)\cap F_x = \{x\}\subseteqq\mathfrak{F}(X)\cap F_1^{-1}$$

and $\mathfrak{F}(X)\subseteqq F_x^1$. Therefore $\tilde{X}_0\subseteqq[\mathfrak{F}(X)]_0$.

Furthermore, one has $\tilde{X}_0\subseteqq F^{-1}\cap F^1$ for $(F^{-1}, F, F^1)\in\mathfrak{F}$ where $x_0\in$ $\in F^{-1}\cap F^1$ and $\mathfrak{F}(X)\subseteqq F^1$.

Moreover $x_0 \notin \tilde{X}_0$. Otherwise $x_0 \in F_{x_0}^{-1} \cap F_{x_0}^1$.

From this fact it follows that $X_0 \cup \{x_0\} \cup \tilde{X}_0 \subseteq F_{x_0}^{-1} \cap F_{x_0}^1$. Then $X_0 \cup \cup \{x_0\} \cup \tilde{X}_0 = \{x_0\}$. Since $X_0 \neq \emptyset$ one obtains $x_0 \in X_0$, a contradiction. Now one has for every element x of $\tilde{X}_0$ the inclusion

$$\{(F^{-1}, F, F^1): (F^{-1}, F, F^1) \in \mathfrak{F}.\ x_0 \in F^{-1} \cap F^1.\ \mathfrak{F}(X) \subseteq F^1.\} \subseteq$$
$$\subseteq \{(F^{-1}, F, F^1): (F^{-1}, F, F^1) \in \mathfrak{F}.\ x \in F^{-1} \cap F^1.\ \mathfrak{F}(X) \subseteq F^1.\}.$$

Since $x_0 \notin F_x$ for $x \in \tilde{X}_0$ one obtains

$$\bigcap \{F\text{: There exists an element } (F^{-1}, F, F^1) \text{ of } P(L) \times P(L) \times P(L) \text{ such that } (F^{-1}, F, F^1) \in \mathfrak{F}.\ x \in F^{-1} \cap F^1.\ \mathfrak{F}(X) \subseteq F^1.\} \subset$$
$$\subset \bigcap \{F\text{: There exists an element } (F^{-1}, F, F^1) \text{ of } P(L) \times P(L) \times P(L) \text{ such that } (F^{-1}, F, F^1) \in \mathfrak{F}.\ x_0 \in F^{-1} \cap F^1.\ \mathfrak{F}(X) \subseteq F^1.\}.$$

Therefore $(\varphi(x), \varphi(x_0)) \in R$ for $x \in \tilde{X}_0$.

Let $x_0 \in [\mathfrak{F}(X)]_0$ such that from $x \in [\mathfrak{F}(X)]_0$ and $(\varphi(x), \varphi(x_0)) \in R$ it follows that $x \in T([\mathfrak{F}(X)]_h)$. Then $x_0 \in [\mathfrak{F}(X)]_h$ or there exists an element $(x_0, \tilde{X}_0)$ of $\mathfrak{Q}$ such that $(\varphi(x), \varphi(x_0)) \in R$ for $x \in \tilde{X}_0$. In the second case one obtains $x \in T([\mathfrak{F}(X)]_h)$ for $x \in \tilde{X}_0$. Therefore $\tilde{X}_0 \subseteq T([\mathfrak{F}(X)]_h)$ and

$$x_0 \in h(\tilde{X}_0) \subseteq h\big(T([\mathfrak{F}(X)]_h)\big) = T([\mathfrak{F}(X)]_h).$$

From lemma 3 one obtains that $[\mathfrak{F}(X)]_0 \subseteq T([\mathfrak{F}(X)]_h)$.

Finally, let x_0 be an element of $\big(\mathfrak{F}(X) - ([\mathfrak{F}(X)]_0 \cup [\mathfrak{F}(X)]_h)\big)$ such that $(x_0, \emptyset) \in \mathfrak{Q}$. Since $\mathfrak{F}(X)$ is $(\mathfrak{Q}, \mathfrak{F})$-compact there exists an element $(x_0, \tilde{X}_0)$ of $\mathfrak{Q}$ such that $\tilde{X}_0 \subseteq [\mathfrak{F}(X)]_0$.

Therefore $x_0 \in h(\tilde{X}_0) \subseteq h([\mathfrak{F}(X)]_0) \subseteq h\big(T([\mathfrak{F}(X)]_h)\big)$ and $x_0 \in T([\mathfrak{F}(X)]_h)$. ▌

Proof of theorem 4. From $[\mathfrak{F}(X)]_T \subseteq \mathfrak{F}(X) \subseteq T([\mathfrak{F}(X)]_T)$ and theorem 1 one obtains $(\mathfrak{F}(X), [\mathfrak{F}(X)]_T) \in \gamma$. ▌

Proof of theorem 5. From

$$[\mathfrak{F}(X)]_T \subseteq [X]_T \subseteq X \subseteq \mathfrak{F}(X) \subseteq T([\mathfrak{F}(X)]_T)$$

and theorem 1 one obtains $(X, [X]_T) \in \gamma$, $(X, \mathfrak{F}(X)) \in \gamma$, $(\mathfrak{F}(X), [\mathfrak{F}(X)]_T) \in \gamma$. ▌

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SPLINE APPROXIMATION IN L^2 SPACE

By

THARWAT FAWZY

Suez Canal University, Ismailia, Egypt

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In the recent years there has been great interest and progress on spline approximation of functions. In most investigations, the functions satisfy certain smoothness conditions. In this paper we construct a cubic spline which converges to a discontinuous function in the L^2 norm if this function is defined and bounded in $[0, 1]$ with that norm.

Let Ω be the interval $[0, 1]$, $L^2(\Omega)$ is the class of those functions $f(x)$ defined on Ω, which are measurable and for which $|f(x)|^2$ is integrable on Ω and have the norm

$$\|f\| = \left(\int_\Omega |f(x)|^2\, dx\right)^{1/2} .$$

Let us divide the interval $[0,1]$ into equal subintervals $[x_k, x_{k+1}]$ and let $h = x_{k+1} - x_k$, $k = 0, 1, \ldots, N-1$. If for any $x \in [0, 1]$,

$$\frac{f(x+0)+f(x-0)}{2} \text{ exists and bounded then we define}$$

(1)
$$y_k = f(x_k) = \frac{f(x_k+0)+f(x_k-0)}{2} .$$

Our spline function approximating $f(x)$ will be given in the following theorem:

Theorem 1. *If Δ denotes the mesh points,*

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_N = 1$$

with $h = x_{k+1} - x_k$ and $k = 0, 1, \ldots, N-1$ then there exists a unique spline function $S_\Delta(x)$, approximating $f(x)$, which has the following properties

(2)
$$S_\Delta(x) \in C^1[0, 1] ,$$

$$S_\Delta(x) = S_k(x) \stackrel{\text{def}}{=} y_k + \frac{y_{k+1}-y_k}{h}(x-x_k) + a_1^{(k)}(x-x_k)^2 + a_2^{(k)}(x-x_k)^3 \tag{3}$$

where $x \in [x_k, x_{k+1}]$ *and* $k = 0, 1, \ldots, N-2,$

$$S_{N-2}(x) \equiv S_{N-1}(x)\,. \tag{4}$$

PROOF. From (2) and (3) it is easy to get

$$a_1^{(k)} + a_2^{(k)} h = 0 \tag{5}$$

and

$$2a_1^{(k)} h^2 + 3a_2^{(k)} h^3 = y_{k+2} - 2y_{k+1} + y_k\,. \tag{6}$$

Equations (5) and (6) have the unique solution

$$a_1^{(k)} = \frac{-1}{h^2}(y_{k+2} - 2y_{k+1} + y_k) \tag{7}$$

and

$$a_2^{(k)} = \frac{1}{h^3}(y_{k+2} - 2y_{k+1} + y_k) \tag{8}$$

and this completes the proof.

THEOREM 2. *For any function* $g(x) \in C^0[0, 1]$, $S_\Delta(x)$ *converges uniformly to* $g(x)$ *as* $h \to 0$ *and for all* $x \in [0, 1]$ *the inequality*

$$|g(x) - S_\Delta(x)| \le 6\omega(h)$$

holds, where $\omega(h)$ *is the modulus of continuity of* $g(x)$.

PROOF. For $x_k \le x \le x_{k+1}$,

$$\begin{aligned}
|g(x) - S_\Delta(x)| &= |g(x) - S_k(x)| \le \\
&\le |g(x) - g(x_k)| + |g_{k+1} - g_k| + 2|g_{k+2} - 2g_{k+1} + g_k| \le \\
&\le |g(x) - g(x_k)| + |g(x_{k+1}) - g(x_k)| + \\
&+ 2\big(|g(x_{k+2}) - g(x_{k+1})| + |g(x_{k+1}) - g(x_k)|\big) \le \\
&\le \omega(h) + \omega(h) + 2\big(\omega(h) + \omega(h)\big) = 6\omega(h)\,.
\end{aligned}$$

THEOREM 3. *Let* $f(x) \in L^2[0, 1]$ *with the norm*

$$\|f\| = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2}.$$

If for any $x \in [0, 1]$, $\frac{1}{2}\big(f(x+0) + f(x-0)\big)$ *exists and bounded, then the spline function* $S_\Delta(x)$ *constructed in* Theorem 1 *converges to* $f(x)$ *as* $h \to 0$ *in the norm defined above.*

PROOF. Let Π_n denotes the set of all polynomials of degree $\le n$. If $P_n(x)$ is the polynomial of best approximation to $f(x)$ in the norm L^2, then for arbitrary $Q_n \in \Pi_n$

$$E_n(f) = \left(\int_0^1 [f(x)-P_n(x)]^2\,dx\right)^{1/2} \le \left(\int_0^1 [f(x)-Q_n(x)]^2\,dx\right)^{1/2}, \tag{9}$$

$$E_0(f) \ge E_1(f) \ge \ldots \quad \text{and} \quad \lim_{n\to\infty} E_n(f) = 0\,.$$

Since for any $V(x)$, the relation

$$\infty > \int_0^1 [V(x)]^2\,dx \approx h\sum_{k=0}^{N} [V(x_k)]^2 \tag{10}$$

is true, then

$$E_n^2 = \int_0^1 [f(x)-P_n(x)]^2\,dx \approx h\sum_{k=0}^{N} (y_k - P_n(x_k))^2 = 0(1) \tag{11}$$

for sufficiently large n.

Let us define for $k = 0, 1, \ldots, N$

$$P_n(x_k) = p_k = \frac{P_n(x_k+0)+P_n(x_k-0)}{2} \tag{12}$$

and for $x_k \le x \le x_{k+1}$, $\bar{S}_\Delta(x) \equiv \bar{S}_k(x)$ denotes the spline function approximating $P_n(x)$ and constructed in the same manner as $S_\Delta(x)$ of Theorem 1, i.e. for $x_k \le x \le x_{k+1}$,

$$\bar{S}_\Delta(x) \equiv S_k(x) = p_k + \frac{p_{k+1}-p_k}{h}(x-x_k) + b_1^{(k)}(x-x_k)^2 + b_2^{(k)}(x-x_k)^3\,, \tag{13}$$

$$b_1^{(k)} = \frac{-1}{h^2}(p_{k+2}-2p_{k+1}+p_k) \tag{14}$$

and

$$b_2^{(k)} = \frac{1}{h^3}(p_{k+2}-2p_{k+1}+p_k). \tag{15}$$

Now we proceed to the proof of the theorem as follows:

$$\int_0^1 [f(x)-S_\Delta(x)]^2\,dx = \tag{16}$$

$$= \int_0^1 \{[f(x)-P_n(x)] + [P_n(x)-\bar{S}_\Delta(x)] + [\bar{S}_\Delta(x)-S_\Delta(x)]\}^2\,dx \le$$

$$\le 2\int_0^1 [f(x)-P_n(x)]^2\,dx + 4\int_0^1 [P_n(x)-\bar{S}_\Delta(x)]^2\,dx +$$

$$+ 4\int_0^1 [\bar{S}_\Delta(x)-S_\Delta(x)]^2\,dx$$

and

$$(17)\qquad \int_0^1 [\bar{S}_\Delta(x)-S_\Delta(x)]^2\,dx = \sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}} \{(p_k-y_k)+$$

$$+\frac{1}{h}(p_{k+1}-p_k-y_{k+1}+y_k)(x-x_k)-$$

$$-\frac{1}{h^2}(p_{k+2}-2p_{k+1}+p_k-y_{k+2}+2y_{k+1}-y_k)(x-x_k)^2+$$

$$+\frac{1}{h^3}(p_{k+2}-2p_{k+1}+p_k-y_{k+2}+2y_{k+1}-y_k)(x-x_k)^3\}^2\,dy =$$

$$=\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}}\left\{A_k+\frac{1}{h}B_k(x-x_k)-\frac{1}{h^2}C_k(x-x_k)^2+\frac{1}{h^3}C_k(x-x_k)^3\right\}^2 dx\le$$

$$\le 2\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}}\left\{A_k+\frac{1}{h}B_k(x-x_k)\right\}^2 dx+$$

$$+2\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}}\left\{-\frac{1}{h^2}C_k(x-x_k)^2+\frac{1}{h^3}C_k(x-x_k)^3\right\}^2 dx\le$$

$$\le 4\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}} A_k^2\,dx+4\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}}\frac{1}{h^2}B_k^2(x-x_k)^2\,dx+$$

$$+4\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}}\frac{1}{h^4}C_k^2(x-x_k)^4\,dx+4\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}}\frac{1}{h^6}C_k^2(x-x_k)^6\,dx=$$

$$=4h\sum_{k=0}^{N-1}\left(A_k^2+\frac{1}{3}B_k^2+\frac{12}{35}C_k^2\right)$$

and from (11) we get,

$$(18)\qquad h\sum_{k=0}^{N-1}A_k^2 = o(1)\,,$$

$$(19)\qquad h\sum_{k=0}^{N-1}B_k^2 = o(1)$$

and

$$(20)\qquad h\sum_{k=0}^{N-1}C_k^2 = o(1)\,.$$

Combining these results of (18), (19) and (20) with the last result of (17) we get,

$$\int_0^1 [\overline{S}_\Delta(x) - S_\Delta(x)]^2\, dx = o(1) \tag{21}$$

for sufficiently large n.

Now consider

$$I = \int_0^1 [P_n(x) - \overline{S}_\Delta(x)]^2\, dx\,. \tag{22}$$

Since $P_n(x)$ is a continuous function, we get from Theorem 2, where $g(x)$ is replaced by $P_n(x)$,

$$|P_n(x) - \overline{S}_\Delta(x)| \leq 6\omega(P, h) \tag{23}$$

where $\omega(P, h)$ is the modulus of continuity of $P_n(x)$, and thus

$$\int_0^1 [P_n(x) - \overline{S}_\Delta(x)]^2\, dx = o(1)\,. \tag{24}$$

From (11), (21) and (24) the result of (16) becomes

$$\int_0^1 [f(x) - \overline{S}_\Delta(x)]^2\, dx = o(1) \tag{25}$$

and this completes the proof.

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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF CERTAIN FUNCTIONAL-DIFFERENTIAL EQUATIONS

By

G. G. HAMEDANI

University of Wisconsin – Milwaukee Milwaukee, Wisconsin U.S.A.

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As pointed out by A. BIELECKI, L. JANOS, and others ([1], [5]) the Banach contraction principle, when applied to the theory of differential equations, provides proofs of existence and uniqueness of solutions only in a local sense. In [2], S. C. CHU and J. B. DIAZ have shown that the contraction principle can be applied to functional equations if the metric of the underlying function space is suitably changed. In [3], W. DERRICK and L. JANOS applied the ideas of CHU and DIAZ to the differential equation $y' = f(x, y)$, where $f(x, y)$ is a continuous function from $(-a, a)\times E^m$ into E^m, $0 < a \leq \infty$, satisfying a certain global Lipschitz condition. In [4], the present author extends the result in [3] to the differential equation $y^{(n)}(x) = f(x, y, y^{(1)}, \ldots, y^{(n-1)})$, where $f(x, y, y^{(1)}, \ldots, y^{(n-1)})$ is continuous on $(-a, a)\times E^m \times \ldots \times E^m$ into E^m, $0 < a < \infty$, satisfying a global Lipschitz condition.

In the present paper we are concerned with the problem of the existence, uniqueness and continuous dependence of solutions of the following functional-differential equation

$$y''(x) = f\big(x, y(h_1(x)),\ y'(h_2(x)),\ u\big), \tag{1}$$

in the interval $J = (-a, a)$, $0 < a \leq \infty$, where h_i, $i = 1, 2$ are continuous on J into J, f is a continuous vector-valued function on $J \times E^m \times E^m \times E$ into E^m satisfying the global Lipschitz condition

$$|f(x, y_1, y_2, u) - f(x, y_1^*, y_2^*, u)| \leq \sum_{i=1}^{2} z_i(x)\,|y_i - y_i^*| \tag{2}$$

for every (y_1, y_2), $(y_1^*, y_2^*) \in E^m \times E^m$, $x \in J$, $u \in E$ and non-negative continuous functions $z_i(x)$, $i = 1, 2$ defined on J such that:

either

(A1) $z_i(x) \geq c_i > 0$, $i = 1, 2$ for all $x \in J$ and $z_i(x)$ non-increasing for $x > 0$ and non-decreasing for $x < 0$;

or

(A2) $z_i(x) \leq k$, $i = 1, 2$ for all $x \in J$.

Here c_1, c_2, k are some constants and $|\cdot|$ denotes the usual norm in E^m.

Following [3], we let $\{I_k \mid j \geq 1\}$ be an increasing family of compact intervals which contain zero and $\bigcup_j I_j = J$. We denote by $c(I_j)$ the Banach space of continuous vector-valued functions $g: I_j \to E^m$ with the norm

$$\|g\|_{(j,\lambda)} = \sup_{x \in I_j} \left\{ \exp\left(-\lambda \left| \int_0^x z_1(t)\, z_2(t)\, dt \right| \right) |g(x)| \right\} \tag{3}$$

if (A1) holds; and with the norm

$$\|g\|_{(j,\lambda)} = \sup_{x \in I_j} \{\exp(-\lambda |x|)\, |g(x)|\} \tag{4}$$

if (A2) holds, where λ is an arbitrary parameter. The Fréchét space $c(J)$ may be topologized by the family of seminorms $\{\|g\|_{(j,\lambda)} \mid j \geq 1\}$. If $\lambda = 0$, the spaces $c(I_j)$ have the usual sup norm $\|\cdot\|_0$ on I_j.

THEOREM 1. *If the function $f(x, y_1, y_2, u)$ satisfies* (2) *and* (A1) *and if*

$$x h_i(x) \geq 0, \quad |h_i(x)| \leq |x|, \quad x \in J, \quad i = 1, 2, \tag{5}$$

then the initial value problem $y(0) = y_0$, $y'(0) = y_1$, has a unique solution y for every $(y_0, y_1) \in E^m \times E^m$, which is given as the limit of successive approximations.

PROOF. Let I be a compact subinterval of J containing zero and for simplicity, denote the norm of $g \in c(I)$ by $\|g\|_\lambda$. From (3), it follows that the norms $\|g\|_\lambda$, for arbitrary real λ, are all equivalent to the norm $\|g\|_0$. The identity

$$\left| \int_0^x \exp\left(\lambda \left| \int_0^t z(s)\, ds \right| \right) z(t)\, dt \right| = \frac{1}{\lambda} \left\{ \exp\left(\lambda \left| \int_0^x z(t)\, dt \right| \right) - 1 \right\} \tag{6}$$

is valid for every $x \in J$, $\lambda > 0$.

We shall reduce our problem, by substitution

$$y(x) = y_0 + x y_1 + \int_0^x \int_0^s g(t)\, dt\, ds$$

to the following equation

$$g(x) = f\left(x, y_0 + h_1(x) y_1 + \int_0^{h_1(x)} \int_0^s g(t)\, dt\, ds, y_1 + \int_0^{h_2(x)} g(s)\, ds, u \right). \tag{7}$$

Let $u \in E$ be fixed. It is obvious that the transformation $\Phi = T(g)$ defined by the right-hand side of (7) maps $c(I)$ continuously into itself. We shall prove that

(8) $$\|Tg_2 - Tg_1\| \le \frac{1}{c\lambda}\left(1+\frac{1}{c^2\lambda}\right)\|g_2-g_1\|_\lambda$$

for all $g_1, g_2 \in c(I)$ and $\lambda > 0$, where $c = \min\{1, c_1, c_2\}$. Using (2) and the definition of $\|\cdot\|_\lambda$ we have:

$$|Tg_2(x) - Tg_1(x)| \le z_1(x)\left|\int_0^{h_1(x)}\int_0^s [g_2(t)-g_1(t)]\,dt\,ds\right| +$$

$$+ z_2(x)\left|\int_0^{h_2(x)} [g_2(s)-g_1(s)]\,ds\right| \le$$

$$\le z_1(x)\left|\int_0^{h_1(x)}\int_0^s |g_2(t)-g_1(t)|\,ds\right| + z_2(x)\left|\int_0^{h_2(x)} |g_2(s)-g_1(s)|\,ds\right| \le$$

$$\le \|g_2-g_1\|_\lambda\left\{z_1(x)\left|\int_0^{h_1(x)}\int_0^s \exp\left(\lambda\left|\int_0^t z_1(u)z_2(u)\,du\right|\right)dt\,ds\right| +\right.$$

$$\left. + z_2(x)\left|\int_0^{h_2(x)} \exp\left(\lambda\left|\int_0^s z_1(u)z_2(u)\,du\right|\right)ds\right|\right\} \le$$

$$\le \|g_2-g_1\|_\lambda\left\{z_1(x)\left|\int_0^x\int_0^s \exp\left(\lambda\left|\int_0^t z_1(u)z_2(u)\,du\right|\right)dt\,ds\right| +\right.$$

$$\left. + z_2(x)\left|\int_0^x \exp\left(\lambda\left|\int_0^s z_1(u)z_2(u)\,du\right|\right)ds\right|\right\} \le$$

$$\le \|g_2-g_1\|_\lambda\left\{z_1(x)\left|\int_0^x \frac{1}{c^2\lambda}\exp\left(\lambda\left|\int_0^s z_1(u)z_2(u)\,du\right|\right)ds\right| +\right.$$

$$\left. + z_2(x)\left|\int_0^x \exp\left(\lambda\left|\int_0^s z_1(u)z_2(u)\,du\right|\right)ds\right\} \le\right.$$

$$\le \|g_2-g_1\|_\lambda\left\{\frac{1}{c^3\lambda}\left|\int_0^x z_1(s)z_2(s)\exp\left(\lambda\left|\int_0^s z_1(u)z_2(u)\,du\right|\right)ds\right| +\right.$$

$$\left. + \frac{1}{c}\left|\int_0^x z_1(s)z_2(s)\exp\left(\lambda\left|\int_0^s z_1(u)z_2(u)\,du\right|\right)ds\right|\right\} \le$$

$$\le \frac{1}{c\lambda}\left(1+\frac{1}{c^2\lambda}\right)\exp\left(\lambda\left|\int_0^x z_1(s)z_2(s)\,ds\right|\right)\|g_2-g_1\|_\lambda,$$

where we have used (5), (6), ((A1) and (6)), and (6) to obtain, the fourth, fifth, sixth, and the seventh inequalities respectively. Thus

$$\|Tg_2-Tg_1\|_\lambda \leq \frac{1}{c\,\lambda}\left(1+\frac{1}{c^2\,\lambda}\right)\|g_2-g_1\|_\lambda\,.$$

Now choose $\lambda>0$ so that $\frac{1}{c\lambda}\left(1+\frac{1}{c^2\lambda}\right)<1$ and apply the classical Banach contraction principle to T and the distance function $\|g_2-g_1\|_\lambda$ to complete the proof.

THEOREM 2. *The conclusion of theorem* 1 *holds under the assumptions* (2), (A2), *and* (5).

PROOF. Using (4) and similar calculations used in the proof of theorem 1, we obtain

$$\|Tg_2-Tg_1\|_\lambda \leq \frac{k}{\lambda}\left(1+\frac{1}{\lambda}\right)\|g_2-g_1\|_\lambda \tag{9}$$

for all $g_1, g_2 \in c(I)$ and $\lambda>0$. To complete the proof we need to choose $\lambda>0$ so that $\frac{k}{\lambda}\left(1+\frac{1}{\lambda}\right)<1$.

Now we consider the problem of continuous dependence of solutions of our problem on a parameter u.

THEOREM 3. *Let the hypotheses of theorem* 1 *be satisfied. If there exist a constant M and a function $G: J \to J$ such that for every $x\in J$, u, $u_1\in E$, $(y_1, y_2)\in \in E^m\times E^m$*

$$|f(x, y_1, y_2, u)-f(x, y_1, y_2, u_1)| \leq G(x)\,|u-u_1| \tag{10}$$

and

$$\sup_{x\in J}\left\{\exp\left(-\lambda\left|\int_0^x z_1(t)\,z_2(t)\,dt\right|\right)G(x)\right\}\leq M\,, \tag{11}$$

then solution $y(x, u)$ of (1) *fulfilling $y(0, u)=y_0$, $y'(0, u)=y_1$ is continuous with respect to the variables (x, u) in $J\times E$.*

PROOF. For $g\in c(I)$ we define the transformation $T_u(g)$ by the right-hand side of the equation (7). From (8) we have

$$\|T_u(g)-T_u(y)\|_\lambda \leq \frac{1}{c\,\lambda}\left(1+\frac{1}{c^2\,\lambda}\right)\|g-y\|_\lambda\,.$$

From the hypotheses we obtain

$$\|T_u(g)-T_{u_1}(g)\|_\lambda \leq M\,|u-u_1|\,.$$

From theorem 1, there exists unique solution $g(x, u)$, $g(\cdot, u)\in c(J)$ such that

$$y(x, u)=y_0+xy_1+\int_0^x\int_0^s g(t, u)\,dt\,ds$$

$$T_u(g(x, u)) = g(x, u), \quad T_{u_1}(g(x, u_1)) = g(x, u_1) \quad \text{for} \quad x \in J .$$

Therefore, we have

$$\|g(x, u) - g(x, u_1)\|_\lambda \le \left(1 - \frac{1}{c\lambda}\left(1 + \frac{1}{c^2\lambda}\right)\right)^{-1} M\,|u - u_1| .$$

Hence y is continuous with respect to two variables $(x, u) \in J \times E$.

THEOREM 4. *The conclusion of theorem* 3 *holds under the assumptions of theorem* 2, (10) *and*

$$\sup_{x \in J} \{\exp(-\lambda |x|)\, G(x)\} \le M . \tag{12}$$

REMARK. The results of this paper can be extended without difficulties to the following equation

$$y^{(n)}(x) = f\Big(x, y(h_1(x)),\ y^{(1)}(h_2(x)), \ldots, y^{(n-1)}(h_n(x)), u\Big),$$

which in turn extend the result in [4].

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ON TWO SCHEMES APPLIED TO MAL'CEV TYPE THEOREMS

By

JAROMÍR DUDA

Brno, Czechoslovakia

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The present paper deals with two relational and two functional characterizations of congruence relations on universal algebras and their applications to varieties of algebras. Following the recent investigations, see e.g. [2], [3], [6], we consider not only congruences but also other sorts of compatible binary relations as: *tolerances* (= compatible symmetric and reflexive relations), *quasiorders* (= compatible transitive and reflexive relations), and *compatible reflexive relations*. In particular, for any algebra $\mathfrak{A}$ and any subset M of $\mathfrak{A}\times\mathfrak{A}$, the symbol

$\Theta(M)$ denotes the congruence on $\mathfrak{A}$ generated by M;

$T(M)$ denotes the tolerance on $\mathfrak{A}$ generated by M;

$Q(M)$ denotes the quasiorder on $\mathfrak{A}$ generated by M; and

$R(M)$ denotes the compatible reflexive relation on A generated by M.

For the sake of brevity, we will write $\Theta(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle)$ instead of $\Theta(\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\})$ and $\Theta(a, b)$ instead of $\Theta(\{\langle a, b\rangle\})$; analogously for tolerances, quasiorders etc. The detailed results on compatible binary relations may be found in [2] and [3]; for other undefined notions we refer the reader to the relevant sections of [8].

It is well-known that the "generated" congruences, i.e. the congruences of the form $\Theta(M)$, $M\subseteq\mathfrak{A}\times\mathfrak{A}$, play an important role in universal algebra and thus the characterization of $\Theta(M)$ known under the name Mal'cev Lemma, see [8; Thm 4, p. 55], is frequently used. This Lemma contains the set-theoretical condition $\{p_i(a_i), p_i(b_i)\} = \{z_{i-1}, z_i\}$ ensuring the symmetry of $\Theta(M)$ and so there appears the question of a suitable description of the mentioned set equality by purely algebraic notions. In [3], the symmetry of $\Theta(a, b)$ is expressed by means of tolerance $T(a, b)$ using the formula

$$(*)\qquad \Theta(a, b) = \bigcup_{n<\omega} T^n(a, b)$$

where $T^n(a, b)$ denotes the relation product $T(a, b) \circ \ldots \circ T(a, b)$ (n-factors, $n \geq 1$) and $T^0(a, b)$ stands for the diagonal on $\mathfrak{A}$. The right side of $(*)$ can be easily expressed in terms of binary algebraic functions; the obtained result characterizing $\Theta(a, b)$ is therefore called the *binary scheme*. In this paper we extend the binary scheme for arbitrary generated congruences, see Theorem 1 below.

As was just noted, the concept of binary scheme has its origin in the symmetry of congruences, however, in many cases this property follows directly from assumptions, e.g. this phenomenon can be found on algebras in an n-permutable variety as was shown by J. HAGEMANN [10; Cor. 4, p. 7]: "*A variety $\boldsymbol{V}$ is n-permutable for some integer $n > 1$ if and only if congruences coincide with quasiorders on any algebra $\mathfrak{A} \in \boldsymbol{V}$*". Clearly, in this situation we consider $Q(M)$ instead of $\Theta(M)$ and thus tolerances in $(*)$ are rewritten by reflexive compatible relations. This simplifies the functional characterization of $\Theta(M)$ since the binary algebraic functions from the binary scheme are replaced by unary algebraic functions, in other words, we get the *unary scheme* characterizing $\Theta(M)$, see Theorem 2 of this paper.

We begin with the following three lemmas. Firstly, we recall the functional description of $T(a, b)$ and $R(a, b)$; for the proof, see [2].

LEMMA 0. *Let a, b be elements of an algebra $\mathfrak{A}$. Then*

(a) $\langle x, y\rangle \in T(a, b)$ *if and only if there exists a binary algebraic function β over $\mathfrak{A}$ such that $x = \beta(a, b)$, $y = \beta(b, a)$; briefly: $\langle x, y\rangle = (\beta\times\beta)(\langle a, b\rangle, \langle b, a\rangle)$.*

(b) $\langle x, y\rangle \in R(a, b)$ *if and only if there exists a unary algebraic function α over $\mathfrak{A}$ such that $x = \alpha(a)$, $y = \alpha(b)$; briefly: $\langle x, y\rangle = (\alpha\times\alpha)(\langle a, b\rangle)$.*

Further, we express $\Theta(M)$ and $Q(M)$ by means of suitable tolerances and compatible reflexive relations, respectively.

LEMMA 1. *Let $\mathfrak{A}$ be an algebra and let M be a subset of $\mathfrak{A}\times\mathfrak{A}$. Then*

$$\Theta(M) = \bigcup\{T(u_1, v_1) \circ \ldots \circ T(u_n, v_n);\ \langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle \in M\}.$$

PROOF. Since the inclusion $T(a, b) \subseteq \Theta(a, b)$ holds for any elements a, b of $\mathfrak{A}$ we have also $T(u_1, v_1) \circ \ldots \circ T(u_n, v_n) \subseteq \Theta(u_1, v_1) \circ \ldots \circ \Theta(u_n, v_n) \subseteq \Theta(M)$ for any $\langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle \in M$ and thus

$$\bigcup\{T(u_1, v_1) \circ \ldots \circ T(u_n, v_n);\ \langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle \in M\} \subseteq \Theta(M).$$

Conversely, denote by U the union

$$\bigcup\{T(u_1, v_1) \circ \ldots \circ T(u_n, v_n);\ \langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle \in M\}.$$

Then it is easily seen that

(i) $M \subseteq U$;
(ii) U is reflexive, symmetric, and transitive binary relation on $\mathfrak{A}$;
(iii) U is a subalgebra of the square $\mathfrak{A}\times\mathfrak{A}$ since it is a directed union of subalgebras of $\mathfrak{A}\times\mathfrak{A}$.

In summary, U is a congruence relation on $\mathfrak{A}$ containing the subset M; hence $\Theta(M) \subseteq U$ and the proof is complete.

An analogous result for quasiorders is given in the following lemma; the proof goes along the same line as that of Lemma 1 and is therefore omitted.

LEMMA 2. *Let $\mathfrak{A}$ be an algebra and let M be a subset of $\mathfrak{A}\times\mathfrak{A}$. Then*

$$Q(M) = \cup\{R(u_1, v_1) \circ \ldots \circ R(u_n, v_n);\ \langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle \in M\}.$$

Now, we are ready to characterize $\Theta(M)$ and $Q(M)$ in terms of algebraic functions; the first statement is a slight generalization of [3; Thm 1].

THEOREM 1. *Let $\mathfrak{A}$ be an algebra and let M be a subset of $\mathfrak{A}\times\mathfrak{A}$. Then the following conditions are equivalent:*

(1) $\langle x, y\rangle \in \Theta(M)$;
(2) $\langle x, y\rangle \in T(u_1, v_1) \circ \ldots \circ T(u_n, v_n)$ for some $\langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle \in M$;
(3) (Binary scheme) *There are elements $u_1, \ldots, u_n, v_1, \ldots, v_n$ of A and binary algebraic functions $\beta_1, \ldots, \beta_n$ over A such that $\langle u_i, v_i\rangle \in M$ for $1 \le i \le n$, and*

$$x = \beta_1(u_1, v_1)$$
$$\beta_i(v_i, u_i) = \beta_{i+1}(u_{i+1}, v_{i+1}) \quad \textit{for} \quad 1 \le i < n$$
$$y = \beta_n(v_n, u_n).$$

PROOF. (1)⇒(2) follows directly from the preceding Lemma 1.

(2)⇒(3). By the definition of relation product, there are elements $a_1, \ldots, a_{n+1}$ of $\mathfrak{A}$ such that

$$x = a_1, \quad y = a_{n+1},$$

and

$$\langle a_i, a_{i+1}\rangle \in T(u_i, v_i) \quad \text{for} \quad 1 \le i \le n.$$

Using Lemma 0 (a) we get binary algebraic functions $\beta_1, \ldots, \beta_n$ over $\mathfrak{A}$ with

$$a_i = \beta_i(u_i, v_i), \quad a_{i+1} = \beta_i(v_i, u_i) \quad \text{for} \quad 1 \le i \le n.$$

In summary, condition (3) follows.

(3)⇒(1). Clearly, $\langle \beta_i(u_i, v_i), \beta_i(v_i, u_i)\rangle \in T(u_i, v_i)$ holds for all $1 \le i \le n$. Combining this with

$$x = \beta_1(u_1, v_1)$$
$$\beta_i(v_i, u_i) = \beta_{i+1}(u_{i+1}, v_{i+1}) \quad \text{for} \quad 1 \le i < n$$
$$y = \beta_n(v_n, u_n)$$

we easily get $\langle x, y\rangle \in T(u_1, v_1) \circ \ldots \circ T(u_n, v_n)$. Moreover, since $\langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle \in M$ we have $T(u_1, v_1) \circ \ldots \circ T(u_n, v_n) \subseteq \Theta(M)$ and thus $\langle x, y\rangle \in \Theta(M)$ completing the proof.

Combining Lemma 0 (b) with Lemma 2 we immediately get an analogous result for $Q(M)$; it will be used under the name unary scheme in the sequel.

THEOREM 2. *Let $\mathfrak{A}$ be an algebra and let M be a subset of $\mathfrak{A}\times\mathfrak{A}$. Then the following conditions are equivalent:*

(1) $\langle x, y\rangle \in Q(M)$;
(2) $\langle x, y\rangle \in R(u_1, v_1) \circ \ldots \circ R(u_n, v_n)$ for some $\langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle \in M$;
(3) (Unary scheme) *There are elements $u_1, \ldots, u_n, v_1, \ldots, v_n$ of A and unary algebraic functions $\alpha_1, \ldots, \alpha_n$ over $\mathfrak{A}$ such that $\langle u_i, v_i\rangle \in M$ for $1 \le i \le n$, and*

$$x = \alpha_1(u_1)$$
$$\alpha_i(v_i) = \alpha_{i+1}(u_{i+1}) \quad \textit{for} \quad 1 \le i < n$$
$$y = \alpha_n(v_n)\,.$$

The proof is similar to that of Theorem 1 and hence is omitted.

We can now apply the binary and unary schemes to regular and weakly regular varieties. This choice has two reasons. Firstly, regular and weakly regular varieties form in fact a very suitable material for direct applications of the preceding theorems and, secondly, the obtained results simplify the former Mal'cev characterizations of regularity and weak regularity, see [1], [4], [5], [6], [7], [9], [10], and [16].

First of all notice that regular varieties were characterized by $(2n+3)$-ary polynomials $\lambda_1, \ldots, \lambda_k$ in [4]. By applying J. Hagemann's theorems:

"*Any regular variety is n-permutable for $n > 1$*", see [10; Thm 3, p. 11], and

"*A variety $\boldsymbol{V}$ is n-permutable for some $n > 1$ if and only if congruences coincide with quasiorders on any $\mathfrak{A} \in \boldsymbol{V}$*", see [10; Cor. 4, p. 7], the $(2n+3)$-ary polynomials $\lambda_1, \ldots, \lambda_k$ were replaced by $(n+3)$-ary polynomials $r_1, \ldots, r_k$ in [6]. However, as was shown in [5], $(2+3)$-ary polynomials $p_1, \ldots, p_k$ are enough to characterize regular varieties. It can be easily seen, that the latter polynomials follow from the binary scheme.

Moreover, J. Hagemann's results entitle to use the unary scheme only and thus regular varieties are described by $(1+3)$-ary polynomials $r_1, \ldots, r_n$ now.

Writing the above considerations in a more precise form we immediately get

THEOREM 3. *For a variety $\boldsymbol{V}$, the following conditions are equivalent:*

(1) *$\boldsymbol{V}$ is regular;*
(2) *For any elements x, y, z of an algebra $\mathfrak{A} \in \boldsymbol{V}$, $Q(x, y) = Q(\langle z, p_1\rangle, \ldots, \langle z, p_m\rangle)$ holds for some elements $p_1, \ldots, p_m$ of $\mathfrak{A}$;*
(3) *There exist ternary polynomials $p_1, \ldots, p_n$ and 4-ary polynomials $r_1, \ldots, r_n$ such that*

$$x = r_1(z, x, y, z)$$
$$r_i(p_i(x, y, z), x, y, z) = r_{i+1}(z, x, y, z) \quad \textit{for} \quad 1 \le i < n$$
$$y = r_n(p_n(x, y, z), x, y, z)$$
$$z = p_i(x, x, z) \quad \textit{for} \quad 1 \le i \le n\,.$$

PROOF. (1)⇒(2). It follows directly from the so-called Hashimoto Lemma, see [12] or [9; Lemma 1, p. 335], that for any elements x, y, z of a regular algebra $\mathfrak{A}$, $\Theta(x, y) = \Theta(\langle z, p_1\rangle, \dots, \langle z, p_m\rangle)$ is satisfied for some elements $p_1, \dots, p_m$ of $\mathfrak{A}$. Further, by J. HAGEMANN [10], congruences may be replaced by quasiorders on any algebra from a regular variety. In summary, condition (2) is proved.

(2)⇒(3). Take $\mathfrak{A} = F_3(x, y, z)$ the free algebra in $\boldsymbol{V}$ with free generators x, y, z and assume (2). Then, by applying the unary scheme to the condition "$\langle x, y\rangle \in Q(\langle z, p_1\rangle, \dots, \langle z, p_m\rangle)$ for some $p_1, \dots, p_m \in F_3(x, y, z)$", the identities of (3) easily follow.

(3)⇒(1). Apparently, the ternary polynomials $p_1, \dots, p_n$ from (3) satisfy: $(z = p_i(x, y, z),\ 1 \le i \le n) \Leftrightarrow x = y$, i.e. we get the well-known B. CSÁKÁNY's criterion for regular varieties, see [4; p. 188]. Consequently, condition (1) holds and the proof is complete.

For varieties with nullary operations, say $c_1, \dots, c_k$, the concept of regularity was generalized to that of weak regularity as follows: A variety $\boldsymbol{V}$ with nullary operations $c_1, \dots, c_k$ is said to be *weakly regular* with respect to $c_1, \dots, c_k$ if for any congruences Θ, Ψ on each $\mathfrak{A} \in \boldsymbol{V}$ the equalities $[c_i]\Theta = [c_i]\Psi$, $1 \le i \le k$, imply $\Theta = \Psi$.

Mal'cev conditions for weakly regular varieties were derived in [1], [6], [7], [9], [10], moreover, [10; Thm 6, p. 14] states that

"*Any weakly regular (with respect to nullary operations $c_1, \dots, c_k$) variety is* $(2n+1)$*-permutable for some* $n \ge 1$" and thus we use the unary scheme in weakly regular varieties too.

For the proof of our next theorem we need the following modification of H. A. THURSTON's result, see [15]. Without loss of generality we will consider varieties with one nullary operation only.

LEMMA 3. *Let $\boldsymbol{V}$ be a variety with nullary operation c. Then the following conditions are equivalent:*

(a) *$\boldsymbol{V}$ is weakly regular with respect to c;*
(b) *A congruence Θ on an algebra $\mathfrak{A} \in \boldsymbol{V}$ is trivial whenever $[c]\Theta$ is a singleton.*

PROOF. The implication (a)⇒(b) is obvious and thus it suffices to prove the converse implication (b)⇒(a): Take a congruence Ψ on an algebra $\mathfrak{A} \in \boldsymbol{V}$. We show that Ψ is uniquely determined by its class $[c]\Psi$, i.e. we prove the equality $\Psi = \Theta([c]\Psi \times [c]\Psi)$. Clearly, $\Psi \supseteq \Theta([c]\Psi \times [c]\Psi)$ and so we can consider the congruence relation $\Psi/\Theta([c]\Psi \times [c]\Psi)$ on the quotient algebra $\mathfrak{A}/\Theta([c]\Psi \times [c]\Psi) \in \boldsymbol{V}$. By the construction, $[c](\Psi/\Theta([c]\Psi \times [c]\Psi))$ is a singleton and thus, using the hypothesis, $\Psi/\Theta([c]\Psi \times [c]\Psi)$ is trivial congruence on $\mathfrak{A}/\Theta([c]\Psi \times [c]\Psi) \in \boldsymbol{V}$; hence $\Psi = \Theta([c]\Psi \times [c]\Psi)$ and the proof is complete.

Now we are ready to prove

THEOREM 4. *Let $\boldsymbol{V}$ be a variety with nullary operation c. Then the following conditions are equivalent:*

(1) *$\boldsymbol{V}$ is weakly regular with respect to c;*
(2) *For any elements x, y of an algebra $\mathfrak{A} \in \boldsymbol{V}$, $Q(x, y) = Q(\langle c, q_1 \rangle, \ldots, \langle c, q_m \rangle)$ holds for some elements $q_1, \ldots, q_m$ of $\mathfrak{A}$;*
(3) *There exist binary polynomials $q_1, \ldots, q_n$ and ternary polynomials $w_1, \ldots, w_n$ such that*

$$x = w_1(c, x, y)$$
$$w_i(q_i(x, y), x, y) = w_{i-1}(c, x, y) \quad \text{for} \quad 1 \leq i < n$$
$$y = w_n(q_n(x, y), x, y)$$
$$c = q_i(x, x) \quad \text{for} \quad 1 \leq i \leq n;$$

(4) *There exist binary polynomials $q_1, \ldots, q_n$ satisfying $\left(c = q_i(x, y), 1 \leq i \leq n\right) \Leftrightarrow x = y$.*

PROOF. (1)⇒(2). Analogously to the proof of Theorem 3 above, this part follows immediately from the Hashimoto Lemma and J. Hagemann's theorems [10; Thm 6, p. 14 and Cor. 4, p. 7].

(2)⇒(3). Choose $\mathfrak{A} = F_2(x, y)$ the free algebra in $\boldsymbol{V}$ with free generators x, y. Then, by applying the unary scheme to the condition "$\langle x, y \rangle \in Q(\langle c, q_1 \rangle, \ldots, \langle c, q_m \rangle)$ for some $q_1, \ldots, q_m \in F_2(x, y)$", the required identities of (3) follow.

(3)⇒(4). Immediate.

(4)⇒(1). Take an arbitrary congruence relation Θ on an algebra $\mathfrak{A} \in \boldsymbol{V}$ and suppose that $[c]\Theta$ is a singleton. Since $\langle q_i(a, b), c \rangle = \langle q_i(a, b), q_i(a, a) \rangle \in \in \Theta$, $1 \leq i \leq n$, for any $\langle a, b \rangle \in \Theta$, we have $q_i(a, b) \in [c]\Theta = \{c\}$, $1 \leq i \leq n$. By assumption, the equalities $c = q_i(a, b)$, $1 \leq i \leq n$, imply $a = b$ proving the triviality of Θ. Lemma 3 completes the proof.

REMARK. Let $\boldsymbol{V}$ be a variety weakly regular with respect to nullary operation c. Then, using Theorem 4 (3), we introduce the ternary polynomials $h_1, \ldots, h_n$ via $h_i(t, u, v) := w_i(q_i(u, v), t, v)$, $1 \leq i \leq n$. Clearly, these polynomials satisfy

$$t = h_1(t, v, v)$$
$$h_i(t, t, v) = h_{i+1}(t, v, v) \quad \text{for} \quad 1 \leq i < n$$
$$v = h_n(t, t, v)$$

proving the $(n+1)$-permutability of $\boldsymbol{V}$, see [10], [11] or [8]. As the same result holds for varieties weakly regular with respect to nullary operations $c_1, \ldots, c_k$, J. Hagemann's theorem [10; Thm 6, p. 14] can be expressed in the following form:

"*Any weakly regular variety (with respect to nullary operations $c_1, \ldots, c_k$) is n-permutable for some $n > 1$*".

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О ПРИМЕНЕНИИ МЕТОДА УСРЕДНЕНИЯ ДЛЯ РЕШЕНИЯ МНОГОТОЧЕЧНЫХ КРАЕВЫХ ЗАДАЧ С НЕЛИНЕЙНЫМ КРАЕВЫМ УСЛОВИЕМ ДЛЯ ОДНОГО КЛАССА ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ СОДЕРЖАЩИХ КРАТНЫЕ ИНТЕГРАЛЫ

Д. Д. БАЙНОВ и С. Д. МИЛУШЕВА

Пловдивский Университет имени П. Хилендарского, Пловдив
и Высший Машинно-Электротехнический Институт имени В. И. Ленина, София

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В настоящей работе обоснован метод усреднения для решения многоточечных краевых задач с нелинейным краевым условием для интегро-дифференциальных уравнений содержащих кратные интегралы.

Рассмотрим систему

$$\dot{x}(t) = \varepsilon X\Big(t, x(t), \int_0^t \varphi_1(t, s_1, x(s_1))\,ds_1, \ldots, \int_0^t \overset{m}{\cdots} \int_0^t \varphi_m(t, s_1, \ldots, s_m, x(s_1), \ldots, x(s_m))\,ds_1 \ldots ds_m\Big) \tag{1}$$

с краевым условием

$$\sum_{i=0}^{N} A_i x(t_i) = \Gamma\big(x(t_0), \ldots, x(t_N), \varepsilon\big), \tag{2}$$

где

$$x, X, \Gamma \in R_n, \quad \varphi_k \in R_{n_k}, \quad k = 1, \ldots, m, \quad A_i = \big(a_{lj}^{(i)}\big)_1^n,$$

$$t_i = \alpha_i T, \quad i = 0, 1, \ldots N, \quad 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_N = 1,$$

$T = L\,\varepsilon^{-1}$, $L = \mathrm{const} > 0$, а $\varepsilon > 0$ — малый параметр.

Пусть существует предел

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X\Big(t, x, \int_0^t \varphi_1(t, s_1, x)\,ds_1, \ldots, \int_0^t \overset{m}{\cdots} \int_0^t \varphi_m(t, s_1, \ldots, s_m, x, \ldots, x)\,ds_1 \ldots ds_m\Big)\,dt = \overline{X}(x). \tag{3}$$

Тогда системе (1) ставим в соответствие усредненную систему

$$\dot{\xi}(t) = \varepsilon \overline{X}(\xi(t)) \tag{4}$$

с краевым условием

$$\sum_{i=0}^{N} A_i \xi(t_i) = \Gamma(\xi(t_0), \ldots, \xi(t_N), \varepsilon). \tag{5}$$

Отметим, что если $x = (x^{(1)}, \ldots, x^{(n)})$ и $A = (a_{ij})_{l,m}$, то по определению

$$\|x\| = \left[\sum_{i=1}^{N} (x^{(i)})^2\right]^{\frac{1}{2}}, \quad \|A\| = \left[\sum_{j=1}^{m} \sum_{i=1}^{l} a_{ij}^2\right]^{\frac{1}{2}}.$$

Справедлива следующая теорема:

Теорема 1. *Пусть:*

1. *Функция* $X(t, x, y_1, \ldots, y_m)$ *определена и непрерывна в области*

$$\Omega^* = \Omega(t, x, y_1, \ldots, y_m) = \Omega(t) \times \Omega(x) \times \Omega(y_1) \times \ldots \times \Omega(y_m),$$

где $\Omega(t) = [0, \infty)$, $\Omega(x)$ — *некоторая открытая область пространства* R_n, $\Omega(y_k) \equiv R_{n_k}$, $k = 1, \ldots, m$.

Функция $\varphi_k(t, s_1, \ldots, s_k, u_1, \ldots, u_k)$, $k = 1, \ldots, m$ *определена и непрерывна в области*

$$\Omega(t) \times \Omega_k = \Omega(t) \times \Omega(s_1, \ldots, s_k, u_1, \ldots, u_k) =$$
$$= \Omega(t) \times \Omega(s_1) \times \ldots \times \Omega(s_k) \times \Omega(u_1) \times \ldots \times \Omega(u_k),$$

где $\Omega(s_j) = [0, \infty)$, $\Omega(u_j) \subseteq R_{n_j}$, $j = 1, \ldots, k$.

Функция $\Gamma(z, \varepsilon)$, $z = (z_0, \ldots, z_N)$ *определена в области*

$$\Omega(z, \varepsilon) = \underbrace{\Omega(x) \times \ldots \times \Omega(x)}_{N+1} \times \Omega(\varepsilon),$$

где $\Omega(\varepsilon) = (0, \varepsilon_1]$, $\varepsilon_1 = \text{const} > 0$.

2. *В соответствующих проекциях области* $\Omega^* \times \Omega_m \times \Omega(z, \varepsilon)$ *функции* $X(t, x, y_1, \ldots, y_m)$, $\varphi_k(t, s_1, \ldots, s_k, u_1, \ldots, u_k)$ *и* $\Gamma(z, \varepsilon)$ *удовлетворяют условиям*

$$\|X(t, x, y_1, \ldots, y_m)\| \leq M,$$

$$\|X(t, x, y_1, \ldots, y_m) - X(t, x', y_1', \ldots, y_m')\| \leq$$
$$\leq \lambda(\|x - x'\| + \|y_1 - y_1'\| + \ldots + \|y_m - y_m'\|),$$

$$\|\varphi_k(t, s_1, \ldots, s_k, u_1, \ldots, u_k) - \varphi_k(t, s_1, \ldots, s_k, u_1', \ldots, u_k')\| \leq$$
$$\leq \mu_k(t, s_1, \ldots, s_k)(\|u_1 - u_1'\| + \ldots + \|u_k - u_k'\|), \quad k = 1, \ldots, m,$$

$$\|\Gamma(z, \varepsilon) - \Gamma(z', \varepsilon)\| \leq \sum_{i=0}^{N} \lambda_i \|z_i - z_i'\|,$$

где M, λ, λ_0 — *положительные постоянные,* λ_i, $i = 1, \ldots, N$ *зависят от* ε, *функция* $b(\varepsilon) = \max_i \lambda_i(\varepsilon)$ *непрерывна при достаточно малых значениях* ε, $\lim_{\varepsilon \to 0} b(\varepsilon) = 0$ *и*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t d\tau \int_0^\tau \overset{k}{\ldots} \int_0^\tau \mu_k(\tau, s_1, \ldots, s_k)\, ds_1 \ldots ds_k = 0, \quad k = 1, \ldots, m.$$

3. *Краевая задача* (1), (2) *имеет единственное непрерывное решение* $x(t)$ *и* $x(t) \in \Omega(x)$ *при* $t \in [0, T]$.

4. *В каждой точке* $x \in \Omega(x)$ *существует предел* (3). *Функция* $\overline{X}(x)$ *в области* $\Omega(x)$ *непрерывна и удовлетворяет условию Липшица*

$$\|\overline{X}(x) - \overline{X}(x')\| \le \nu \|x - x'\|, \quad \nu = \text{const}.$$

5. *Краевая задача* (4), (5) *имеет единственное, непрерывное решение* $\xi(t)$ *и* $\xi(t) \in \Omega(x)$ *при* $t \in [0, T]$.

6. *Матрица* A_0 *постоянная и* $\det A_0 \neq 0$.

7. *Матрицы* A_i, $i = 1, \ldots, N$ *зависят от* ε, *функция*

$$d(\varepsilon) = \max_i \|A_i(\varepsilon)\|$$

непрерывна при достаточно малых значениях ε *и* $\lim_{\varepsilon \to 0} d(\varepsilon) = 0$.

8. *Выполнено неравенство*

$$\left\| \left(\sum_{i=0}^{N} A_i \right)^{-1} \right\| \sum_{i=0}^{N} \lambda_i < 1.$$

Тогда, для любых $\eta > 0$ *и* $L > 0$ *можно указать такое число* $\varepsilon_0 > 0$, *что при* $0 < \varepsilon \le \varepsilon_0$ *на отрезке* $0 \le t \le T$ *выполнялось неравенство*

$$\|x(t) - \xi(t)\| < \eta.$$

Доказательство. Для решений краевых задач (1), (2) и (4), (5), в силу условий теоремы 1, выполнены равенства

$$x(t) = x_0 + \varepsilon \int_0^t X\Big(\tau, x(\tau), \int_0^\tau \varphi_1(\tau, s_1, x(s_1))\, ds_1, \ldots,$$

(6)

$$\int_0^\tau \overset{m}{\ldots} \int_0^\tau \varphi_m(\tau, s_1, \ldots, s_m, x(s_1), \ldots, x(s_m))\, ds_1 \ldots ds_m\Big) d\tau,$$

(7) $$\xi(t) = \xi_0 + \varepsilon \int_0^t \overline{X}(\xi(\tau))\, d\tau,$$

(8) $$\sum_{i=0}^{N} A_i(x_0 + \varepsilon \beta_i) = \Gamma(x_0, x_0 + \varepsilon \beta_1, \ldots, x_0 + \varepsilon \beta_N, \varepsilon),$$

(9) $$\sum_{i=0}^{N} A_i(\xi_0+\varepsilon\bar\beta_i) = \Gamma(\xi_0, \xi_0+\varepsilon\bar\beta_1, \dots, \xi_0+\varepsilon\bar\beta_N, \varepsilon),$$

где

$$x_0 = x(0), \quad \xi_0 = \xi(0),$$

$$\beta_i = \int_0^{t_i} X\Bigl(\tau, x(\tau), \int_0^{\tau}\varphi_1(\tau, s_1, x(s_1))\,ds_1, \dots,$$

$$\int_0^{\tau}\overset{m}{\dots}\int_0^{\tau}\varphi_m(\tau, s_1, \dots, s_m, x(s_1), \dots, x(s_m))\,ds_1\dots ds_m\Bigr)\,d\tau,$$

$$\bar\beta_i = \int_0^{t_i}\bar X(\xi(\tau))\,d\tau, \quad i = 0, 1, \dots, N.$$

Вычитая (7) из (6) получаем

(10) $$\|x(t)-\xi(t)\| \le \|x_0-\xi_0\| + \varepsilon\Bigl\|\int_0^t\Bigl[X\Bigl(\tau, x(\tau), \int_0^{\tau}\varphi_1(\tau, s_1, x(s_1))\,ds_1, \dots,$$
$$\int_0^{\tau}\overset{m}{\dots}\int_0^{\tau}\varphi_m(\tau, s_1, \dots, s_m, x(s_1), \dots, x(s_m))\,ds_1\dots ds_m\Bigr) - \bar X(\xi(\tau))\Bigr]\,d\tau\Bigr\|.$$

Для второго слагаемого в правой стороне неравенства (10) справедлива оценка

$$\varepsilon\Bigl\|\int_0^t\Bigl[X\Bigl(\tau, x(\tau), \int_0^{\tau}\varphi_1(\tau, s_1, x(s_1))\,ds_1, \dots,$$

$$\int_0^{\tau}\overset{m}{\dots}\int_0^{\tau}\varphi_m(\tau, s_1, \dots, s_m, x(s_1), \dots, x(s_m)) - \bar X(\xi(\tau))\Bigr]\,d\tau\Bigr\| \le$$

$$\le \delta(\varepsilon) + \varepsilon\Bigl\|\int_0^t\Bigl[X\Bigl(\tau, \xi(\tau), \int_0^{\tau}\varphi_1(\tau, s_1, \xi(\tau))\,ds_1, \dots,$$

$$\int_0^{\tau}\overset{m}{\dots}\int_0^{\tau}\varphi_m(\tau, s_1, \dots, s_m, \xi(\tau), \dots, \xi(\tau))\,ds_1\dots ds_m\Bigr) - \bar X(\xi(\tau))\Bigr]\,d\tau\Bigr\| +$$

$$+\varepsilon\lambda\int_0^t\Bigl\{\|x(\tau)-\xi(\tau)\| + \int_0^{\tau}\mu_1(\tau, s)\,\|x(s_1)-\xi(s_1)\|\,ds_1 + \dots +$$

$$+\int_0^{\tau}\overset{m}{\dots}\int_0^{\tau}\mu_m(\tau, s_1, \dots, s_m)\,[\|x(s_1)-\xi(s_1)\| + \dots +$$

$$+\|x(s_m)-\xi(s_m)\|]\,ds_1\dots ds_m\Bigr\}\,d\tau,$$

где

$$\delta(\varepsilon) = \lambda LM \sum_{k=1}^{m} k\,\delta_k(\varepsilon), \quad \delta_k(\varepsilon) = \sup_{0\le l\le L} l\,\gamma_k(l\,\varepsilon^{-1}),$$

$$\gamma_k(t) = \frac{1}{t}\int_0^t d\tau \int_0^\tau \overset{k}{\dots} \int_0^\tau \mu_k(\tau, s_1, \dots, s_k)\,ds_1\dots ds_k, \quad k = 1, \dots, m.$$

В силу условий теоремы 1, как в [1], можно показать, что любому положительному числу ε можно поставить в соответствие функцию $a(\varepsilon, p)$, $(a(\varepsilon, p)\to 0$ при $p\to\infty$ и $\varepsilon\to 0)$, где p натуральное число для которой на отрезке $0\le t\le T$ выполняется неравенство

$$\varepsilon\left\|\int_0^t\left[X\left(\tau, \xi(\tau), \int_0^\tau \varphi_1(\tau, s_1, \xi(\tau))\,ds_1, \dots,\right.\right.\right.$$
$$\left.\int_0^\tau \overset{m}{\dots}\int_0^\tau \varphi_m(\tau, s_1, \dots, s_m, \xi(\tau), \dots, \xi(\tau))\,ds_1\dots ds_m\right)-$$
$$\left.\left.-\bar{X}(\xi(\tau))\right]d\tau\right\|\le a(\varepsilon, p).$$

Таким образом, для всех $t\in[0, T]$ выполняется неравенство

$$\varepsilon\left\|\int_0^t\left[X\left(\tau, x(\tau), \int_0^\tau \varphi_1(\tau, s_1, x(s_1))\,ds_1, \dots,\right.\right.\right.$$
$$\left.\left.\left.\int_0^\tau \overset{m}{\dots}\int_0^\tau \varphi_m(\tau, s_1, \dots, s_m, x(s_1), \dots, x(s_m))\,ds_1\dots ds_m\right)-\bar{X}(\xi(\tau))\right]d\tau\right\|\le$$

(11)

$$\le\delta(\varepsilon)+a(\varepsilon, p)+\varepsilon\lambda\int_0^t\Big\{\|x(\tau)-\xi(\tau)\|+\int_0^\tau \mu_1(\tau, s_1)\,\|x(s_1)-\xi(s_1)\|\,ds_1+\dots+$$
$$+\int_0^\tau \overset{m}{\dots}\int_0^\tau \mu_m(\tau, s_1, \dots, s_m)\,[\|x(s_1)-\xi(s_1)\|+\dots+$$
$$+\|x(s_m)-\xi(s_m)\|]\,ds_1\dots ds_m\Big\}\,d\tau.$$

Вычитая (9) из (8), после некоторых выкладок, получаем неравенство

$$\left(1-\left\|\left(\sum_{i=0}^N A_i\right)^{-1}\right\|\sum_{i=0}^N \lambda_i\right)\|x_0-\xi_0\|\le$$
$$\le\varepsilon\left\|\left(\sum_{i=0}^N A_i\right)^{-1}\right\|\sum_{i=1}^N\left(\|A_i(\varepsilon)\|+\lambda_i(\varepsilon)\right)\|\beta_i-\bar{\beta}_i\|.$$

В силу условий теоремы, из последнего неравенства следует

$$\|x_0-\xi_0\| \leq \varepsilon\, h(\varepsilon) \sum_{i=1}^{N} \|\beta_i-\overline{\beta}_i\| , \tag{12}$$

где

$$h(\varepsilon) = \big(b(\varepsilon)+d(\varepsilon)\big) \left(1-\left\|\left(\sum_{i=0}^{N} A_i\right)^{-1}\right\| \sum_{i=0}^{N} \lambda_i\right)^{-1} \left\|\left(\sum_{i=0}^{N} A_i\right)^{-1}\right\|.$$

Имея ввиду (10)–(12) находим

$$\|x(t)-\xi(t)\| \leq c(\varepsilon, p)\left\{1+h(\varepsilon)\left[N+\lambda\left(\alpha L+N\,\delta^*(\varepsilon)\right) \sup_{0\leq t\leq T} \frac{\|x(t)-\xi(t)\|}{c(\varepsilon, p)}\right]\right\}+$$

$$+\varepsilon\lambda \int_0^t \left\{\|x(\tau)-\xi(\tau)\| + \int_0^\tau \mu_1(\tau, s)\,\|x(s_1)-\xi(s_1)\|\,ds_1+\ldots+\right. \tag{13}$$

$$+\int_0^\tau \overset{m}{\ldots} \int_0^\tau \mu_m(\tau, s_1, \ldots, s_m)\,[\|x(s_1)-\xi(s_1)\|+\ldots+$$

$$\left.+\|x(s_m)-\xi(s_m)\|]\,ds_1\ldots ds_m\right\} d\tau ,$$

где

$$c(\varepsilon, p) = a(\varepsilon, p)+\delta(\varepsilon), \quad \delta^*(\varepsilon) = \sum_{k=1}^{m} k\,\delta_k(\varepsilon), \quad \alpha = \sum_{i=1}^{N} \alpha_i .$$

Положим

$$c(\varepsilon, p)\,u(t) = x(t)-\xi(t)$$

и введем обозначение

$$\|u\|_T = \sup_{0\leq t\leq T} \|u(t)\| .$$

Тогда (13) принимает вид

$$\|u(t)\| \leq 1+h(\varepsilon)\,[N+\lambda\left(\alpha L+N\,\delta^*(\varepsilon)\right)\|u\|_T]+\varepsilon\lambda \int_0^t \left\{\|u(\tau)\|+\right.$$

(14)

$$+\int_0^\tau \mu_1(\tau, s_1)\,\|u(s_1)\|\,ds_1+\ldots+\int_0^\tau \overset{m}{\ldots} \int_0^\tau \mu_m(\tau, s_1, \ldots, s_m)\,[u(s_1)\|+\ldots+$$

$$\left.+\|u(s_m)\|]\,ds_1\ldots ds_m\right\} d\tau .$$

Правую часть неравенства (14) обозначим через $w(t)$.

При $s \le t$ имеет место неравенство $w(s) \le w(t)$ и

$$\frac{w'(t)}{w(t)} = \varepsilon\lambda\left\{\frac{\|u(t)\|}{w(t)} + \int_0^t \mu_1(t, s_1)\frac{\|u(s_1)\|}{w(t)}ds_1 + \ldots + \right.$$

$$\left. + \int_0^t \overset{m}{\ldots} \int_0^t \mu_m(t, s_1, \ldots, s_m)\left[\frac{\|u(s_1)\|}{w(t)} + \ldots + \frac{\|u(s_m)\|}{w(t)}\right]ds_1 \ldots ds_m\right\} \le$$

(15)

$$\le \varepsilon\lambda\left[1 + \int_0^t \mu_1(t, s_1)\,ds_1 + \ldots + m\int_0^t \overset{m}{\ldots} \int_0^t \mu_m(t, s_1, \ldots, s_m)\,ds_1 \ldots ds_m\right].$$

Интегрируя последнее неравенство в пределах от 0 до t получаем

$$w(t) \le w(0)\exp\left\{\varepsilon\lambda\int_0^t\left[1 + \int_0^\tau \mu_1(\tau, s_1)\,ds_1 + \ldots + \right.\right.$$

$$\left.\left. + m\int_0^\tau \overset{m}{\ldots} \int_0^\tau \mu_m(\tau, s_1, \ldots, s_m)\,ds_1 \ldots ds_m\right]d\tau\right\},$$

откуда следует, что

$$\|u(t)\| \le \{1 + h(\varepsilon)\,[N + \lambda(\alpha L + N\,\delta^*(\varepsilon))\,\|u\|_T]\}\exp\left\{\varepsilon\lambda\int_0^t\left[1 + \right.\right.$$

$$\left.\left. + \int_0^\tau \mu_1(\tau, s_1)\,ds_1 + \ldots + m\int_0^\tau \overset{m}{\ldots} \int_0^\tau \mu_m(\tau, s_1, \ldots, s_m)\,ds_1 \ldots ds_m\right]d\tau\right\}$$

и

(16) $$\|u\|_T \le \{1 + h(\varepsilon)\,[N + \lambda(\alpha L + N\,\delta^*(\varepsilon))\,\|u\|_T]\}\exp\{\lambda(L + \delta^*(\varepsilon))\}.$$

Так как $\lim\limits_{\varepsilon\to 0} h(\varepsilon) = 0$, то существует число $\varepsilon_2 > 0$ такое, что при $\varepsilon \in (0, \varepsilon_2]$ выполняется неравенство

$$\lambda(\alpha L + N\,\delta^*(\varepsilon))\,h(\varepsilon)\exp\{\lambda(L + \delta^*(\varepsilon))\} < 1.$$

Тогда из (16) получаем

$$\|u\|_T \le \frac{(1 + Nh(\varepsilon))\exp\{\lambda(L + \delta^*(\varepsilon))\}}{1 - \lambda(\alpha L + N\,\delta^*(\varepsilon))\,h(\varepsilon)\exp\{\lambda(L + \delta^*(\varepsilon))\}} = A,$$

где A — положительное число, т. е.

$$\|x(t) - \xi(t)\| \le A\,c(\varepsilon, p).$$

Выберем p и ε_3 таким образом, чтобы при $\varepsilon \leq \varepsilon_3$ выполнялось неравенство $A\,c(\varepsilon, p) < \eta$.

Тогда из этого неравенства при $\varepsilon_0 = \min\limits_{i=1,2,3} \varepsilon_i$ следует утверждение теоремы 1.

Замечание. Надо отметить, что метод усреднения для решения начальных задач для интегро-дифференциальных уравнений содержащих кратные интегралы был обоснован в работах [2] – [4].

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КРАЕВАЯ ЗАДАЧА ДЛЯ ФУНКЦИОНАЛЬНО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ НЕЙТРАЛЬНОГО ТИПА

Н. Г. КАЗАКОВА и Д. Д. БАЙНОВ

Пловдивский Университет имени П. Хилендарского, Пловдив

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В последние годы в теории дифференциальных уравнений с отклоняющимся аргументом большое внимание отделяется функционально-дифференциальным уравнениям нейтрального типа. Трудности, возникающие при исследовании уравнений нейтрального типа традиционными методами связаны с тем, что рассматриваемые здесь операторы как правило не вполне непрерывны. Применение теории уплотняющих операторов [1] разрешает ряд вопросов, поставленных для таких уравнений [2]–[7].

В работе рассматривается краевая задача для функционально-дифференциальных уравнений нейтрального типа с линейным краевым условием. При естественных ограничениях доказано, что если у рассматриваемой задачи существует решение, то и у возмущенной задачи при малом значении параметра существует тоже решение. При доказательстве изпользуются понятия мера некомпактности [1], вращение уплотняющих операторов [1], принцип неподвижной точки для уплотняющих операторов [1] и основная методика из [2].

Рассмотрим задачу о существовании решений краевой задачи

$$\dot{x}(t) = f(\mu, t, x_t, \dot{x}_t), \quad t\in[0, T]$$

$$x(t) = \varphi(t), \quad t\in[-h, 0] \tag{1}$$

$$\sum_{i=1}^{N} \alpha_i x(t_i) = 0\,,$$

где $\mu\in[0, 1]$, оператор f действует из $[0, 1]\times R^1\times C[-h, 0]\times C[-h, 0]$ в R^n; под x_t понимается функция $x_t(s) = x(t+s)$, $s\in[-h, 0]$; аналогичный смысл имеет $\dot{x}_t$; α_i-реальные постоянные, $\alpha_1\neq 0$, $0 = t_1 < t_2 < \ldots < t_N = T$ фиксированные точки; $\varphi(0) = 0$, $\varphi\in C^1[-h, 0]$.

Предположим, что выполнены следующие условия (A):

1. Оператор f непрерывен.

2. $\sum_{i=1}^{N} \alpha_i t_i \neq 0, \quad \sum_{i=1}^{N} \alpha_i = 0.$

3. $\dot{\varphi}_-(0) = f(\mu, 0, \varphi, \dot{\varphi})$ для любого $\mu \in [0, 1]$.

Пусть C^1_α пространство непрерывно дифференцируемых функций $x: [0, T] \to R^n$ с нормой $\|x\|_{C^1_\alpha} = \|x\|_C + \|\dot{x}\|_C$, таких, что $\sum_{i=1}^{N} \alpha_i x(t_i) = 0$;

$$C^1_{\alpha_0} = \{x \in C^1_\alpha : x(0) = 0,\ \dot{x}(0) = \dot{\varphi}(0)\}.$$

Для любых $\mu \in [0, 1]$, $x \in C^1_{\alpha_0}$ определим функцию $J(\mu, x): R^1 \to R^n$ следующим образом:

$$J(\mu, x)(t) = \int_0^t f(\mu, s, x^*_s, \dot{x}^*_s)\, ds -$$

$$-\left[\left(\sum_{i=1}^{N} \alpha_i t_i\right)^{-1} \cdot t + \alpha_1^{-1}\right] \sum_{i=1}^{N} \alpha_i \int_0^{t_i} f(\mu, s, x^*_s, \dot{x}^*_s)\, ds,$$

(2)

$$x^*(t) = \begin{cases} \varphi(t), & t \in [-h, 0] \\ x(t), & t \in [0, T] \end{cases}, \quad \dot{x}^*(t) = \begin{cases} \dot{\varphi}(t), & t \in [-h, 0] \\ \dot{x}(t), & t \in [0, T] \end{cases}$$

и рассмотрим операторное уравнение

(3) $$x = J(\mu, x), \quad x \in C^1_{\alpha_0}.$$

Лемма 1. При выполнении условия (А) оператор $J(\mu, x)$ действует из $[0, 1] \times C^1_{\alpha_0}$ в C^1_α и непрерывен.

Доказательство. Пусть $\mu \in [0, 1]$, $x \in C^1_{\alpha_0}$. Очевидно, $f(\mu, t, x^*_t, \dot{x}^*_t)$ непрерывно зависит от t. Следовательно, правая часть соотношения (2) имеет смысл и является непрерывно дифференцируемой функцией, причем

$$\frac{d}{dt}[J(\mu, x)](t) = f(\mu, t, x^*_t, \dot{x}^*_t) -$$

$$-\left[\sum_{i=1}^{N} \alpha_i t_i\right]^{-1} \sum_{i=1}^{N} \alpha_i \int_0^{t_i} f(\mu, s, x^*_s, \dot{x}^*_s)\, ds.$$

Кроме того, $\sum_{i=1}^{N} \alpha_i J(\mu, x)(t_i) = 0$. Действительно,

$$\sum_{i=1}^{N} \alpha_i J(\mu, x)(t_i) = \sum_{i=1}^{N} \alpha_i \int_0^{t_i} f\, ds -$$

$$-\sum_{i=1}^{N}\alpha_i\left[\frac{1}{\sum\limits_{i=1}^{N}\alpha_i t_i}t_i+\frac{1}{\alpha_i}\right]\sum_{i=1}^{N}\alpha_i\int_0^{t_i} f\,ds=\sum_{i=1}^{N}\alpha_i\int_0^{t_i} f\,ds-$$

$$-\left[\frac{\sum\limits_{i=1}^{N}\alpha_i t_i}{\sum\limits_{i=1}^{N}\alpha_i t_i}+\frac{\sum\limits_{i=1}^{N}\alpha_i}{\alpha_1}\right]\sum_{i=1}^{N}\alpha_i\int_0^{t_i} f\,ds=0$$

в силу предположения $\sum\limits_{i=1}^{N}\alpha_i=0$.

Итак, $J:[0, 1]\times C'_{\alpha_0}\to C^1_{\alpha}$.

Докажем непрерывность оператора $J(\mu, x)$. Для этой цели достаточно проверить, что из $\|x_k-x\|_{C^1_\alpha}\xrightarrow[(k\to\infty)]{}0$, $\mu_k\xrightarrow[(k\to\infty)]{}\mu$ следует

$$\|J(\mu_k, x_k)(0)-J(\mu, x)(0)\|\xrightarrow[(k\to\infty)]{}0$$

$$\max_{t\in[0, T]}\|[J(\mu_k, x_k)]'(t)-[J(\mu, x)]'(t)\|\xrightarrow[(k\to\infty)]{}0.$$

В силу (4) это будет доказано, если установим, что при

$$\|x_k-x\|_{C^1_\alpha}\to 0,\quad \mu_k\to\mu$$

$$\max_{t\in[0, T]}\|f(\mu_k, t, (x_k^*)_t, (\dot{x}_k^*)_t)-f(\mu, t, x_t^*, \dot{x}_t^*)\|\to 0.$$

Последнее соотношение легко выводится из непрерывности оператора f.

Лемма 2. При выполнении условия (А) функция $x:[0, T]\to R^n$ является решением краевой задачи (1) тогда и только тогда, когда $x\in C^1_{\alpha_0}$, $x=J(\mu, x)$.

Доказательство. Пусть x удовлетворяет (1). Очевидно

$$x(t)=\int_0^t f(\mu, s, x_s^*, \dot{x}_s^*)\,ds$$

и

$$\sum_{i=1}^{N}\alpha_i x(t_i)=\sum_{i=1}^{N}\alpha_i\int_0^{t_i} f(\mu, s, x_s^*, \dot{x}_s^*)\,ds.$$

Следовательно

$$x(t)=\int_0^t f(\mu, s, x_s^*, \dot{x}_s^*)\,ds-\left[\frac{1}{\sum\limits_{i=1}^{N}\alpha_i t_i}t+\frac{1}{\alpha_1}\right]\sum_{i=1}^{N}\alpha_i\int_0^{t_i} f\,ds,$$

т. е.

$$x\in C^1_{\alpha_0}, \quad x = J(\mu, x).$$

Пусть теперь

$$x\in C^1_{\alpha_0}, \quad x = J(\mu, x).$$

Тогда

$$x(t) = \int_0^t f(\mu, s, x_s^*, \dot{x}_s^*)\,ds - \left[\frac{1}{\sum\limits_{i=1}^N \alpha_i t_i} t + \frac{1}{\alpha_1}\right] \sum_{i=1}^N \alpha_i \int_0^{t_i} f\,ds,$$

и следовательно

$$\sum_{i=2}^N \alpha_i x(t_i) = \sum_{i=2}^N \alpha_i \int_0^{t_i} f\,ds - \sum_{i=2}^N \alpha_i \left[\frac{1}{\sum\limits_{i=1}^N \alpha_i t_i} t_i + \frac{1}{\alpha_1}\right] \sum_{i=1}^N \alpha_i \int_0^{t_i} f\,ds =$$

$$= \sum_{i=1}^N \alpha_i \int_0^{t_i} f\,ds - \left[\frac{\sum\limits_{i=2}^N \alpha_i t_i}{\sum\limits_{i=1}^N \alpha_i t_i} + \frac{\sum\limits_{i=2}^N \alpha_i}{\alpha_1}\right] \sum_{i=1}^N \alpha_i \int_0^{t_i} f\,ds =$$

$$= \left[1 - \frac{\sum\limits_{i=1}^N \alpha_i t_i}{\sum\limits_{i=1}^N \alpha_i t_i} - \frac{(-\alpha_1)}{\alpha_1}\right] \sum_{i=1}^N \alpha_i \int_0^{t_i} f\,ds = \sum_{i=1}^N \alpha_i \int_0^{t_i} f\,ds,$$

т. е.,

$$\sum_{i=1}^N \alpha_i \int_0^{t_i} f\,ds = \sum_{i=2}^N \alpha_i x(t_i) = \sum_{i=1}^N \alpha_i x(t_i) = 0.$$

Итак,

$$x(t) = \int_0^t f(\mu, s, x_s^*, \dot{x}_s^*)\,ds, \quad \sum_{i=1}^N \alpha_i x(t_i) = 0.$$

Другими словами, x является решением задачи (I).

Определим функцию $\psi(\Omega) = \chi(\Omega')$ на множестве всех ограниченных подмножеств Ω пространства C^1_α, где $\Omega' = \{\dot{x} : x\in\Omega\}$, а χ-мера некомпактности Хаусдорфа в пространстве C. Несложная проверка показывает, что ψ есть мера некомпактности.

Лемма 3. Пусть выполнено условие (A). Пусть оператор f удовлетворяет условию Липшица с постоянной $K<1$ по последнему аргументу. Пусть кроме того множество $M\subset C^1_{\alpha_0}$ ограничено.

Тогда оператор $J:[0,1]\times M\to C^1_\alpha$ ψ-уплотняет.

Доказательство. Пусть $\Omega\subset M$. Множество $J([0,1]\times\Omega)$ ограничено в C^1_α. Действительно, в силу (2) нам достаточно доказать ограниченность множества

$$\{f(\mu, t, x_t^*, \dot{x}_t^*) : \mu\in[0,1],\ t\in[0,T],\ x\in\Omega\}.$$

Так как оператор f непрерывен на $[0,1]\times R^1\times C[-h,0]\times C[-h,0]$, а множество

$$\{(\mu, t, x_t^*) : \mu\in[0,1],\ t\in[0,T],\ x\in\Omega\}$$

относительно компактно в $[0,1]\times R^1\times C[-h,0]$, то существует постоянная $\mathcal{N}_\Omega$ такая, что

$$\|f(\mu, t, x_t^*, 0)\|\le\mathcal{N}_\Omega\ (\mu\in[0,1],\ t\in[0,T],\ x\in\Omega)\ .$$

Поэтому

$$\|f(\mu, t, x_t^*, \dot{x}_t^*)\|\le\|f(\mu, t, x_t^*, \dot{x}_t^*)-f(\mu, t, x_t^*, 0)\|+$$
$$+\|f(\mu, t, x_t^*, 0)\|\le K\,\mathcal{M}_\Omega+\mathcal{N}_\Omega\,,$$

где $\mathcal{M}_\Omega$ мажоранта норм элементов $x\in\Omega$ в C^1_α.

Пусть множество $\overline{\Omega}$ не компактно в C^1_α. Нам нужно доказать, что

$$\Psi[J([0,1]\times\Omega)]<\Psi[\Omega]\,,$$

т. е.

$$\chi[(J([0,1]\times\Omega))']<\chi[\Omega']\,.$$

Рассмотрим оператор

$$F(\mu, u, y)(t)=f(\mu, t, u+(Y_y)_t, y_t)\,,$$

где

$$(Y_y)(t)=\int\limits_0^t y(s)\,ds,\quad u=y(0).$$

Оператор F χ-уплотняет, (см. [1], [2]).

Положим

$$\Omega_0=\{x(0):x\in\Omega\}\,,$$

$$Q=\left\{\frac{1}{\sum\limits_{i=1}^N\alpha_i t_i}\sum_{i=1}^N\alpha_i\int\limits_0^{t_i} f(\mu, s, x_s^*, \dot{x}_s^*)\,ds : x\in\Omega,\ \mu\in[0,1]\right\}.$$

Из (4) нетрудно видеть, что

$$[J([0,1]\times\Omega)]'\subset F([0,1]\times\Omega_0\times\Omega')-Q\,.$$

Поэтому в силу свойства уплотнения оператора F, вполне ограниченности множества Q и того, что множество Ω' не вполне ограничено, получаем

$$\chi\big((J([0,1]\times\Omega))'\big)\le\chi(F([0,1]\times\Omega_0\times\Omega'))+\chi(Q)<\chi(Q')\,.$$

Этим лемма 3 доказана.

Теорема 1. *Пусть выполнено условие* (A). *Пусть оператор f удовлетворяет условию Липшица по последней переменной с постоянной $K<1$.*

Пусть кроме того задача

$$\dot{x}(t) = f(0, t, x_t, \dot{x}_t), \quad t\in[0, T],$$
$$x(t) = \varphi(t), \quad t\in[-h, 0],$$
$$\sum_{i=1}^{N} \alpha_i x(t_i) = 0$$

имеет решение x и при некотором $R>0$ оператор $J(0, .)$ не имеет на границе $\partial B(x, R)$ сферы $B(x, R)$ неподвижных точек, причем вращение $\gamma(I-J(0, .), \partial B(x, R))\neq 0$.

Тогда при достаточно малых μ задача (1) *имеет хотя бы одного решения.*

Доказательство. Докажем сначала, что при условий теоремы 1 оператор $J(\mu, .)$ не имеет на $\partial B(x, R)$ неподвижных точек при малом μ. Допустим, что это не так. Тогда существуют последовательности $\{x_n\}$ и $\{\mu_n\}$, $x_n\in\partial B(x, R)$, $x_n = J(\mu_n, x_n)$ и $\mu_n\to 0$. В силу уплотнения оператора $J(\mu, .)$ имеем оценку

$$\Psi(\{x_n\}) = \Psi(\{J(\mu_n, x_n)\}) \leq \Psi(\{J([0, 1]\times\{x_n\})\}) < \Psi(\{x_n\}).$$

Следовательно множество $\{x_n\}$ компактно и без ограничения общности можно считать, что $x_n\to x_0$, $x_0\in\partial B(x, R)$.

Перейдем к пределу при $n\to\infty$ в равенстве $x_n = J(\mu_n, x_n)$.

Тогда получаем, что $x_0 = J(0, x_0)$, в чем и противоречие.

В силу леммы 3 гомотопные уплотняющие векторные поля $I-J(0, .)$ и $I-J(\mu, .)$ не имеют на границе сферы радиусом R при малых μ неподвижных точек и следовательно

$$\gamma(I-J(0, .), \partial B(x, R)) = \gamma(I-J(\mu, .), \partial B(x, R)).$$

С другой стороны в силу условия теоремы $\gamma(I-J(0, .), \partial B(x, R))\neq$ $\neq 0$. Следовательно оператор $J(\mu, .)$ имеет в $B(x, R)$ неподвижную точку, которая в силу леммы 2 является решением задачи (1).

Теорема доказана.

Рассмотрим оператор

$$f(\mu, t, x_t, \dot{x}_t) = \mathcal{A}(t, x_t, \dot{x}_t) + \mu\mathcal{B}(t, x_t, \dot{x}_t),$$

где оператор $\mathcal{A}$ действует из $R^1\times C[-h, 0]\times C[-h, 0]$ в R^n, оператор $\mathcal{B}$ действует из $R^1\times C[-h, 0]\times C[-h, 0]$ в R^n, μ-параметр.

Теорема 2. *Пусть выполнены следующие условия:*

1. *Оператор $\mathcal{A}$ непрерывен, аддитивен и однороден по второму и третьему аргументу.*

2. *Оператор $\mathcal{B}$ непрерывен и удовлетворяет условию Липшица по третьему аргументу.*

3. $\sum_{i=1}^{N} \alpha_i t_i \neq 0, \quad \sum_{i=1}^{N} \alpha_i = 0$.

4. $\|\mathcal{A}(t, u, v)\| \leq d \|u\|_C + K \|v\|_C$ $(K < 1)$ *для любого* $u, v \in C[-h, 0]$.

5. *Линейная краевая задача*

$$\dot{x}(t) = \mathcal{A}(t, x_t, \dot{x}_t), \quad t \in [0, T],$$

$$x(t) = \varphi(t), \quad t \in [-h, 0],$$

$$\sum_{i=1}^{N} \alpha_i x(t) = 0$$

имеет решение x *и при некотором* $R > 0$ *оператор* $J(0, .)$ *не имеет на границе* $\partial B(x, R)$ *сферы* $B(x, R)$ *неподвижных точек, причем вращение* $\gamma(I - J(0, .), \partial B(x, R)) \neq 0$.

6. $\dot{\varphi}_{-}(0) = \mathcal{A}(0, \varphi, \dot{\varphi}) + \mu \mathcal{B}(0, \varphi, \dot{\varphi})$.

Тогда при достаточно малых μ *возмущенная задача*

$$\dot{x}(t) = \mathcal{A}(t, x_t, \dot{x}_t) + \mu \mathcal{B}(t, x_t, \dot{x}_t), \quad t \in [0, T],$$

$$x(t) = \varphi(t), \quad t \in [-h, 0],$$

$$\sum_{i=1}^{N} \alpha_i x(t_i) = 0$$

имеет хотя бы одного решения.

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APPLICATION OF THE GRADIENT METHOD TO THE SOLUTION OF BOUNDARY VALUE PROBLEMS FOR A SELF-ADJOINT ORDINARY DIFFERENTIAL EQUATION

By

A. SHAMANDY

II. Department of Analysis, L. Eötvös University, Budapest

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Let $I=[a, b]$ be finite closed interval, $C^{(k)}(I)$ the set of all complex-valued functions which are k-times continuous differentiable in the interval I.

Let $W^{(m)}(I)$ — for simplicity $W^{(m)}$ — denote the set of all complex-valued functions $u: I\to\mathbf{C}$, which are defined on the interval I, and for which $u\in C^{(m-1)}(I)$ and $u^{(m-1)}$ is absolutely continuous on I (and hence $u^{(m)}$ exists a.e.) further $u^{(m)}\in L_2(I)$. In the space $W^{(m)}$ we defined the following scalar product

$$\langle u, v\rangle_{W^{(m)}} = \sum_{k=0}^{m} \langle u^{(k)}, v^{(k)}\rangle = \sum_{k=0}^{m} \int_a^b u^{(k)}\overline{v^{(k)}}\,dx \tag{1}$$

where $u, v\in W^{(m)}$.

The norm induced by this scalar product is

$$\|u\|_{W^{(m)}} = \left\{\sum_{k=0}^{m} \|u^{(k)}\|_{L_2}^2\right\}^{\frac{1}{2}} = \left\{\sum_{k=0}^{m} \int_a^b |u^{(k)}|^2\,dx\right\}^{\frac{1}{2}}. \tag{2}$$

The above function space $W^{(m)}$ with the scalar product (1) is called Sobolev space. It is known that this space is a Hilbert space. Obviously $W^{(0)}=L_2$.

Let N be a given natural number, let the functions $P_0, P_1, \ldots, P_n$ be defined on the interval I and satisfy the restrictions

1. $P_k\in C^{(k)}(I)$, $(k=0, 1, \ldots, N)$,
2. $P_k(x)\geq 0$ for every $x\in I$, $(k=0, 1, \ldots, N-1)$,
3. There exists a constant $m>0$ such that for all $x\in I$

$$P_N(x)\geq m.$$

Consider the following equality

$$(3)\qquad Au = \sum_{k=0}^{N} (-1)^k \frac{d^k}{dx^k}\left(P_k \frac{d^k u}{dx^k}\right) = f.$$

We will define the domain of definition of the operator A in the following way:

Introduce the notation $W_0^{(2N)}$ for the set of all functions $u \in W^{(2N)}$ satisfying the boundary conditions

$$(4)\qquad \begin{aligned} u(a) &= u'(a) = \dots = u^{(N-1)}(a) = \\ &= u(b) = u'(b) = \dots = u^{(N-1)}(b) = 0 \end{aligned}$$

and define

$$(5)\qquad \mathcal{D}(A) = W_0^{(2N)}.$$

It is easy to see that $\mathcal{D}(A)$ is a dense subspace of the Hilbert space $L_2(I)$. We see that for all $u \in \mathcal{D}(A)$ implies $Au \in L_2(I)$. Therefore A is a linear differential operator of order $2N$ defined on the dense subspace $\mathcal{D}(A) \subset L_2(I)$ and A maps into $L_2(I)$.

Now, we can show that the differential operator A defined in (3) is linear, strictly positive definite and self-adjoint in the Hilbert space $L_2(I)$ for which

$$\mathcal{R}(A) = L_2(I).$$

An easy computation shows that for arbitrary $u, v \in \mathcal{D}(A)$

$$(6)\qquad \langle Au, v\rangle = \int_a^b \sum_{k=0}^{N} P_k \frac{d^k u}{dx^k}\, \overline{\frac{d^k v}{dx^k}}\, dx.$$

Thus, in the case $v = u$ we are led to

$$(7)\qquad \langle Au, u\rangle = \int_a^b \sum_{k=0}^{N} P_k \left|\frac{d^k u}{dx^k}\right|^2 dx.$$

From (3) we have

$$(8)\qquad Au = \sum_{k=0}^{2N} f_k \frac{d^k u}{dx^k}, \quad f_{2N} = (-1)^N P_N$$

thus

$$(9)\qquad \langle Au, v\rangle = \langle u, Av\rangle$$

and since $\mathcal{D}(A)$ is a dense subspace of the Hilbert space $L_2(I)$ the equation (9) means that A is a symmetric operator.

Since $P_k \geq 0$, $(k = 0, 1, \dots, N-1)$ and $P_N(x) \geq m$, $x \in I$ it follows from (7) that for any $u \in \mathcal{D}(A)$

$$(10)\qquad \langle Au, u\rangle \geq m\, \|u^{(N)}\|_{L_2}^2.$$

It is easy to see that

$$\|u^{(N)}\|_{L^2} \geq \frac{2^N}{(b-a)^{2N}} \|u\|^2_{L_2}. \tag{11}$$

From (10) and (11) we have

$$\langle Au, u\rangle \geq p\, \|u\|^2_{L_2} \quad (u \in \mathcal{D}(A))$$

where

$$p = \frac{m\, 2^N}{(b-a)^{2N}}.$$

This means that A is strictly positive definite.

Now we will show that $\mathcal{R}(A) = L_2(I)$. Let $C^{(2N)}$ be the set of all functions $u \in C^{(2N)}$ satisfying the boundary condition (4).

We will show that there exist constants α and β $(0 < \alpha < \beta < +\infty)$ such that for every $u \in C_0^{(2N)}$

$$\alpha\, \|u\|_{W^{(2N)}} \leq \|Au\|_{L_2} \leq \beta\, \|u\|_{W^{(2N)}} \tag{12}$$

where

$$\|u\|_{W^{(2N)}} = \left\{ \sum_{k=0}^{2N} \|u^{(k)}\|^2_{L_2} \right\}^{\frac{1}{2}}.$$

Let $u \in C_0^{(2N)}(I)$ be an arbitrary function. The relation (8) gives

$$\|Au\|_{L_2} \leq \sum_{k=0}^{2N} \|f_k\, u^{(k)}\|_{L_2} \leq \sum_{k=0}^{2N} M_k\, \|u^{(k)}\|_{L_2}$$

where

$$M_k = \max |f_k|, \quad (k = 0, 1, \ldots, 2N).$$

Introduce the notation $\beta = \left\{ \sum_{k=0}^{2N} M_k^2 \right\}^{\frac{1}{2}}$ and apply the Cauchy-Schwartz inequality. Then we obtain

$$\|Au\|_{L_2} \leq \beta\, \|u\|_{2N}. \tag{13}$$

Let $u \in C_0^{(2)}(I)$ and take $f = Au$. Then it is clear that $f \in C(I)$. Consider the following boundary value problem with the above function f.

$$\sum_{k=0}^{2N} f_k \frac{d^k v}{dx^k} = f, \tag{14}$$

$$v^{(k)}(a) = v^{(k)}(b) = 0 \quad (k = 0, 1, \ldots, N-1).$$

The choice of the function f has been in such a way that the function $u \in C_0^{(2N)}(I)$ is a solution of the boundary value problem (4). On the other hand, it is well-known (see [1]) that the above solution of the boundary value problem (4) has the following form

(15)
$$u(x) = \int_b^b G(x, y) f(y)\, dy, \quad (x \in I)$$

where G is the Green's function of the linear differential operator A. This Green's function exists because the operator A is strictly positive definite and in this case the equation $Av = 0$ has only the trivial solution in the space $C^{(2N)}(I)$. From the property of the Green's function it is easy to prove that the derivative of the function (15) may be written in the following form

$$u^{(k)}(x) = \int_a^b \frac{\partial^k G(x, y)}{\partial x^k} f(y)\, dy, \quad (k = 0, 1, \ldots, 2N-1),$$

$$u^{(2N)}(x) = \int_a^b \frac{\partial^{2N} G(x, y)}{\partial x^{2N}} f(y)\, dy + \frac{1}{f_{2N}(x)} f(x),$$

since $|f_{2N}(x)| = P_N(x) \geq m > 0$, ($m$ is a constant).

Applying the Cauchy-Schwartz inequality for the above equations we obtain

(16)
$$\|u^{(k)}\|_{L_2} \leq \alpha_k \|f\|_{L_2}, \quad (k = 0, 1, \ldots, 2N-1),$$

(17)
$$\|u^{(2N)}\|_{L_2} \leq \left(\alpha_{2N} + \frac{1}{m}\right) \|f\|_{L_2}$$

where

$$\alpha_k = \left\{\int_a^b \int_a^b \left|\frac{\partial^k G(x, y)}{\partial x^k}\right|^2 dx\, dy\right\}^{\frac{1}{2}}, \quad (k = 0, 1, \ldots, 2N).$$

With the substitution

$$\frac{1}{\alpha} = \sqrt{\sum_{k=0}^{2N-1} \alpha_k^2 + \left(\alpha_{2N} + \frac{1}{m}\right)^2},$$

in the inequalities (16), (17) we are led to

$$\|u\|_{2N} = \left\{\sum_{k=0}^{2N} \|u^{(k)}\|_{L_2}\right\}^{\frac{1}{2}} \leq \frac{1}{\alpha} \|f\|_{L_2}.$$

Hence, taking into account that $f = Au$, we obtain

(18)
$$\alpha \|u\|_{2N} \leq \|Au\|_{L_2}.$$

Let $f \in L_2(I)$ be an arbitrary function. Since $C(I)$ is dense in the Hilbert space $L_2(I)$ there exists a sequence of functions $(f_n) \subset C(I)$ for which $f_n \to f$ in L_2-norm.

Consider the sequence

$$u_n(x) = \int_a^b G(x, y) f_n(y)\, dy, \quad (x \in I)$$

then according to the consideration, we have

$$u_n \in C_0^{2N}(I) \subset W_0^{(2N)}$$

and

$$Au_n = f_n \quad (n = 1, 2, \ldots).$$

The application of inequality (12) gives

$$\|u_n - u_m\|_{W^{(2N)}} \le \frac{1}{\alpha} \|A(u_n - u_m)\|_{L_2} = \frac{1}{\alpha} \|f_n - f_m\| \to 0.$$

This means that the sequence of the functions (u_n) is a Cauchy sequence in the space $W_0^{(2N)}(I)$ with respect to the norm $\|\cdot\|_{2N}$. But by the completeness of $W_0^{(2N)}(I)$ there exists a function $u^* \in W_0^{(2N)}(I)$ such that

$$\|u - u^*\|_{2N} \to 0.$$

Thus, we see that the inequality (17) is satisfied by the functions belonging to the space $C_0^{(2N)}(I)$ and it is also satisfied by the elements of the space $H_0^{(2N)}(I)$. Therefore

$$\|f_n - Au^*\|_{L_2} \le \beta \|u_n - u^*\|_{2N} \to 0.$$

From this it follows that $f_n \to Au^*$ in the L_2-norm. On the other hand $f_n \to f$ in L_2-norm, and hence $Au^* = f$. Thus we have proved that for each function $f \in L_2(I)$ there exists a function $u^* \in H_0^{(2N)}(I)$ with $Au^* = f$ and, finally, we have the desired relation

$$\mathcal{R}(A) = L_2(I).$$

Since $W_0^{(2N)}(I) \subset L_2(I)$, and $\mathcal{R}(A) = L_2(I)$, we have the following situation.

The differential operator A defined on the function space $W_0^{(2N)}$ is a self-adjoint operator in the H-space L_2, $\mathcal{R}(A) = L_2(I)$.

Consider the energetic functional

$$\mathcal{E}(u) = \langle Au, u \rangle - 2\,\mathrm{Re}\,\langle u, f \rangle, \quad (u \in W_0^{(2N)}) \tag{19}$$

corresponding to the equation (3) for which the minimum point is the same as the solution u^* of equation (3).

It will be shown now that the gradient method discussed in [7] can be applied to determine the minimum point of the functional $\mathcal{E}$. To be able to do it we have to find a strictly positive definite operator B in the Hilbert space $L_2(I)$ for which

$$\mathcal{D}(B) = \mathcal{D}(A) = W_0^{(2N)}.$$

We need also constants

$$0 < m \le M < +\infty$$

having the property that for every $u \in \mathcal{D}(A)$ the estimates

$$m\langle Bu, u\rangle \le \langle Au, u\rangle \le M\langle Bu, u\rangle$$

hold.

Consider the differential operator B defined by the equation

$$B(u) = (-1)^N \frac{d^{2N} u}{dx^{2N}}, \quad (u \in W_0^{(2N)}). \tag{20}$$

Note that B is a special case of the differential operator (3), with $P_0 = P_1 = \ldots = P_{N-1} = 0$ and $P_N = 1$. Thus B is also strictly positive definite, self-adjoint operator, and

$$\langle Bu, u\rangle = \int_a^b \left|\frac{d^N u}{dx^N}\right|^2 dx = \|u^{(N)}\|_{L_2}^2. \tag{21}$$

Further, we also have

$$m\|u^{(N)}\|_{L_2}^2 \le \langle Au, u\rangle \le M\|u^{(N)}\|_{L_2}^2 \tag{22}$$

where $m > 0$ is a constant satisfying

$$P_N(x) \ge m, \quad (x \in I)$$

and

$$M = \sum_{k=0}^{N} M_k \frac{(b-a)^{2(N-k)}}{2^{N-k}}, \quad M_k = \max_I P_k.$$

It follows from (21), (22) that for every $u \in W_0^{(2N)}$ we have

$$m\langle Bu, u\rangle \le \langle Au, u\rangle \le M\langle Bu, u\rangle.$$

Now we can apply the gradient method described in [7]. Choose an arbitrary initial function $u_0 \in W_0^{(2N)}$ say $u = 0$. Assume that having applied the gradient method we have obtained the $(n-1)^{\text{th}}$ approximation $u_{n-1} \in W_0^{(2N)}$ of the minimum point of the functional $\mathcal{E}$. Introduce the notation

$$f_n = Au_{n-1} - f.$$

Obviously $f_n \in L_2(I)$. Consider the differential equation

$$Bv = f_n,$$

where B is the differential operator given by (20). It is obvious that this differential equation has a unique solution $v_n \in W_0^{(2N)}$, which can be obtained by elementary methods. Using the function v_n we can compute

$$t_n = -\frac{\langle Bv_n, v_n\rangle}{\langle Au_n, u_n\rangle}.$$

From (6) and (7), it follows that

$$t_n = -\frac{\int_a^b |v_n^{(N)}|^2 dx}{\sum_{k=0}^{N} \int_a^b P_k |v_n^{(k)}|^2 dx}. \tag{23}$$

The n^{th} approximation of the function u^* is

$$u_n = u_{n-1} + t_n v_n$$

and we continue the algorithm. From [7] we also have the speed of the convergence

$$\|u_n - u^*\|_B \leq \frac{1}{\sqrt{mp}} \|Au_0 - f\|_{L_2} q^n . \tag{24}$$

Where $q = \dfrac{M-m}{M+m}$, and $p > 0$ is a constant satisfying the inequality

$$\langle Bu, u\rangle \geq p\,\|u\|_{L_2}^{2}, \quad (u \in W_0^{(2N)}) .$$

Since

$$\|u\|_B = \sqrt{\langle Bu, u\rangle} = \|u^{(N)}\|_{L_2}, \quad (u \in W_0^{(2N)})$$

the above algorithm gives that the n^{th} derivative of the sequence u_n in L_2-norm tends to the n^{th} derivative of the solution u^* of equation (3).

We will show for $k = 0, 1, \ldots, N-1$, that

$$u_n^{(k)} \to u^{*(k)} \quad \text{uniformly in} \quad I .$$

Define in the function space $W_0^{(2N)}$ with the norm $\|u\|_N'$ such that

$$\|u\|_N' = \sum_{k=0}^{N-1} \max_I |u^{(k)}| + \|u^{(N)}\|_{L_2} .$$

It is easy to prove that there eixsts a constant $K > 0$ such that for any $u \in W_0^{(N)}$, we have

$$\|u\|_N' \leq K \|u\|_B . \tag{25}$$

By using inequality (24) and (25) we get

$$\|u_n - u^*\|_N' \leq K \|u_n - u^*\|_B \leq \frac{K}{\sqrt{mp}} \|Au_0 - f\|_{L_2} q^n .$$

From this, for $k = 0, 1, \ldots, N-1$, it follows that

$$\max_I |u_n^{(k)} - u^{*(k)}| \leq \frac{K}{\sqrt{mp}} \|Au_0 - f\|_{L_2} q^n .$$

Further, we can conclude that for $k = 0, 1, \ldots, N-1$ $u_n^{(k)} \to u^{*(k)}$ uniformly in I.

Corollary. There exists a constant $K_0 > 0$ such that

$$|u^*(x) - u_n(x)| \leq K_0 q^n, \quad (n = 0, 1, \ldots) .$$

Remark. By simple calculation we can prove that

$$\|\cdot\|_N' \cong \|\cdot\|_B .$$

Summing up, we proved the following results.

SUMMARY. By applying the gradient method discussed in [7], we obtain the sequence $(u_n) \subset W_0^{(2N)}$ that converges to the solution u^* such that $u_n^{(k)} \to u^{*(k)}$ uniformly in I, $k = 0, 1, \ldots, N-1$ and

$$u_n^{(N)} \to u^{*(N)} \quad \text{in} \quad L_2\text{-norm}.$$

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APPLICATION OF THE GRADIENT METHOD TO THE SOLUTION OF THE EQUATION $Ax = f$ IN THE CASE OF UNBOUNDED OPERATORS

By

A. SHAMANDY

II. Department of Analysis, L. Eötvös University, Budapest

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1. Introduction

There are two distinct advantages of the gradient method. The first and foremost is that we need not select a closed system in the image space of the operator. Secondly, it is not necessary to solve a large system of linear equations as in Ritz's method or least square method in order to calculate the n^{th} approximation of any real valued function. One more advantage of the method is its rapid convergence.

The method can be successfully applied in normed spaces to determine the minimum point of any real valued function. The essence of the method is the following:

Let F be a real-valued function defined in the subspace $\mathcal{D}(F)$ of the normed space X. In order to determine the minimum point x^*, let us choose an arbitrary vector $x_0 \in \mathcal{D}(F)$ and a vector $e \in \mathcal{D}(F)$, for which $\|e\| =$ constant holds. Compute the derivative of the function F in the direction e at the point x_0, i.e.

$$F_e'(x_0) = \lim_{t \to 0} \frac{F(x_0 + te) - F(x_0)}{t}.$$

Assume that the derivative $F_e'(x_0)$ takes the largest value in the direction e_1, with $\|e_1\| =$ constant. Consider the function $g : \mathbf{R} \to \mathbf{R}$ defined by the equation

$$g(t) = F(x_0 + te_1), \quad (t \in \mathbf{R}).$$

Calculate the minimum of the function $g(t)$, and assume that this minimum is taken at the point t_1. Put

$$x_1 = x_0 + t_1 e_1$$

and repeat the above construction with x_1.

If we continue this, by iteration we obtain a sequence

$$x_n = x_{n-1} + t_n e_n,$$

where e_n is the direction for which $F_{e_n}'(x_n)$ takes the maximum, and t_n is the minimum point of the function

$$g(t) = F(x_{n-1} + te_n), \quad (t \in \mathbf{R}).$$

2. The case of bounded operator

L. V. Kantorovich (see [2]) applied the gradient method to the solution of the equation

$$(1) \qquad Ax = f,$$

where $A: H \to H$, is bounded, linear, self-adjoint, and strictly positive definite operator in Hilbert space H, $f \in H$ is a given vector. From this assumption on the operator A we know that there exists a unique solution $x^* \in H$ for the equation (1), which is the same as the minimum point of the energetic functional

$$\mathcal{E}(x) = \langle Ax, x \rangle - 2\,\mathrm{Re}\,\langle x, f \rangle, \quad x \in H$$

(see [5]).

Apply the gradient method to obtain the approximation of the minimum point of the energetic functional $\mathcal{E}$. Begin with an arbitrary chosen vector $x_0 \in H$ and assume that we have obtained the $(n-1)^{\text{th}}$ approximation of the minimum point x^* of the functional $\mathcal{E}$ such that $x_{n-1} \in H$ where

$$e_n = Ax_{n-1} - f,$$

and

$$t_n = -\frac{\|e_n\|^2}{\langle Ae_n, e_n \rangle}.$$

Then the n^{th} approximation of the minimum point x^* is

$$x_n = x_{n-1} + t_n e_n.$$

Kantorovich proved that the sequence (x_n) converges to the minimum point x^* of the functional $\mathcal{E}$, and obtained the error estimation

$$(2) \qquad \|x_n - x^*\| \leq \frac{1}{\sqrt{m}} \|Ax_0 - f\| q^n$$

where $q = M - m/(M+m)$, m and M are lower and upper bounds of the operator A, for which

$$0 < m \leq M < +\infty$$

thus $q < 1$.

3. The case of unbounded operator

According to several authors the great disadvantage of the gradient method is that it can be applied only for the bounded operators. The main theme of this paper is, however, to apply this method successfully for the case of unbounded, linear, self-adjoint, and strictly positive definite operators such as the differential operators.

Let A be an unbounded, linear, self-adjoint and strictly positive definite operator defined on a dense subspace of the Hilbert space H, and let $f \in H$

be a given vector. For this case the gradient method may not be applicable directly for the approximate solution of the equation

$$(3) \qquad Ax = f .$$

From the assumption of the operator A, we have $\mathcal{R}(A) = H$ there is because for all $f \in H$, there exists a unique solution for equation (3).

If A is not self-adjoint operator, but only symmetric and strictly positive definite, then we know ([5]) such that operator A can be extended to self-adjoint one, for which

$$\mathcal{R}(A) = H .$$

Although, the lower bound of A is positive and the upper is not finite (i.e. $m_A > 0$ and $M_A = +\infty$). Thus the convergence of the algorithm is not guaranteed by inequality (2).

In the following we assume that there exists a self-adjoint, strictly positive definite and unbounded operator B associated with the unbounded operator A for which

$$\mathcal{D}(B) = \mathcal{D}(A) = H_0 ,$$

where H_0 is a dense subspace in the space H. Furthermore, we can find constants m and M with

$$0 < m \leq M < +\infty ,$$

and are such that for any $x_0 \in H_0$, we have

$$(4) \qquad m\langle Bx, x\rangle \leq \langle Ax, x\rangle \leq M\langle Bx, x\rangle .$$

Note that such an unbounded operator B and the constants m and M always exist. Thus above construction does not restrict the generality.

We will prove that by applying the gradient method one can develop a constructive procedure (algorithm) for the solution of the equation (3). However, at each step, we shall have to solve an equation of type

$$(5) \qquad Bx = g ,$$

where $g \in H$ is a given known vector. We may choose the operator B in such a way that the solution of the equation (5) is much simpler than the solution of the original equation (3).

Introduce the scalar product

$$[x, y]_B = \langle Bx, y\rangle, \quad (x, y \in H_0)$$

and the norm induced by this scalar product is

$$\|x\|_B = \sqrt{\langle Bx, x\rangle} .$$

Let H_B be the Hilbert space obtained by the completion of the Euclidean space $(H_0, [x, y]_B)$. We can see that

$$H_0 \subset H_B \subset H .$$

Consider the bilinear symmetric functional $\mathcal{A}$, which is defined by the equality

$$\mathcal{A}(x, y) = \langle Ax, y \rangle, \quad (x, y \in H_0).$$

From (4) we have

$$m \|x\|_B^2 \leq \mathcal{A}(x, y) \leq M \|x\|_B^2, \quad (x \in H_0). \tag{6}$$

This shows $\mathcal{A}$ is bounded symmetric bilinear functional onto the space H_0. The functional $\mathcal{A}$ and the inequality (6) can be extended on the whole space H_B. However, by Riesz's theorem, there exists a continuous, linear self-adjoint operator

$$A^\circ : H_B \to H_B$$

for which

$$\mathcal{A}(x, y) = [A^\circ x, y], \quad (x, y \in H_B),$$

such that

$$m \|x\|_B^2 \leq [A^\circ x, y] \leq M \|x\|_B^2. \tag{7}$$

It follows that A° is a strictly positive definite bounded operator on the Hilbert space H_B.

Since $\mathcal{R}(B) = H$, then it is easy to prove that

$$A^\circ|_{H_0} = B^{-1} A,$$

thus we see that the original equation (3) is equivalent to the equation

$$A^\circ x = g \tag{8}$$

with $g = B^{-1} f$.

Let us recall the essential difference between the two equations (3) and (8). The left side of equation (3) involves an unbounded linear, self-adjoint and strictly positive definite operator which is defined on a dense subspace of the Hilbert space H. However, the left hand side of equation (6) involves a bounded, linear self-adjoint and strictly positive operator A° which is defined on the whole space H_B.

Now the gradient method can be applied to obtain the solution of (8) in the energetic space H_B. Begin with an arbitrary vector $x_0 \in H_B$ and apply the gradient method. According to KANTOROVICH [2], we can obtain the following approximate sequence

$$x_n = x_{n-1} + t_n u_n \quad (n = 1, 2, \dots)$$
$$u_n = A^\circ x_{n-1} - g$$

where

$$t_n = \frac{-\|u_n\|_B^2}{[A^\circ u_n, u_n]}.$$

Applying the error estimations obtained by Kantorovich, we get

$$\|x_n - x^*\|_B \leq \frac{1}{\sqrt{m_{A^\circ}}} \|A^\circ x_0 - g\|_B \left(\frac{M_{A^\circ} - m_{A^\circ}}{M_{A^\circ} + m_{A^\circ}} \right)^n. \tag{9}$$

We can transform the above equations and the error estimate in such a way that the operator A° does not play any role since $A^\circ|_{H_0} = B^{-1}A$ and $g = B^{-1}f$, we have

$$Bu_n = Ax_{n-1} - f.$$

The vector u_n is the solution of the equation

$$Bu_n = f_n$$

where $f_n = Ax_{n-1} - f$. Moreover, it is easy to see that

$$t_n = \frac{\langle Bu_n, u_n\rangle}{\langle Au_n, u_n\rangle}.$$

Finally, we investigate the speed of the convergence of the sequence (x_n). We know the inequality

$$m\,\|x\|_B^2 \le [A^\circ x, x] \le M\,\|x\|_B^2 \quad (x \in H_B)$$

and also the bounds of the bounded self-adjoint operator $A^\circ : H_B \to H_B$, namely $m_{A^\circ} \ge m$ and $M_{A^\circ} \ge M$. It is easy to prove that

(10) $$\frac{M_{A^\circ} - m_{A^\circ}}{M_{A^\circ} + m_{A^\circ}} \le \frac{M-m}{M+m}$$

and $\|A^\circ x_0 - g\|_B \le \frac{1}{\sqrt{p}}\|A^\circ x_0 - f\|$ where $p > 0$ is constant for which

(11) $$\langle Bx, x\rangle \ge p\,\|x\|_B^2, \quad (x \in H_0).$$

Using (9), (10) and the relation $\frac{1}{\sqrt{m_{A^\circ}}} \le \frac{1}{\sqrt{m}}$, we have

(12) $$\|x_n - x^*\|_B \le \frac{1}{\sqrt{mp}}\,\|Ax_0 - f\|\left(\frac{M-m}{M+m}\right)^n.$$

Since for arbitrary $x \in H_B$ we have $\|x\| < \frac{1}{\sqrt{p}}\|x\|_B$ for the original norm in the space we obtain the following estimation

(13) $$\|x_n - x^*\| \le \frac{1}{p\sqrt{m}}\,\|Ax_0 - f\|\,q^n, \quad q = \frac{M-m}{M+m}.$$

The above algorithm can be summed up in the following theorem.

THEOREM. *Begin with an arbitrary vector $x_0 \in H$, and assume that we obtained x_{n-1} the $(n-1)^{\text{th}}$ approximation of the solution x^* of equation* (3), *where*

$$f_n = Ax_{n-1} - f$$

let u_n be the solution of the equation

$$Bu = f_n$$

then the n^{th} approximation of the solution x^* is

$$x_n = x_{n-1} + t_n u_n$$

where $t_n = \dfrac{\langle Bu_n, u_n\rangle}{\langle Au_n, u_n\rangle}$. *And we can obtain the error estimation* (12) *and* (13).

STATEMENT 1. From the above discussion we see that for applying the gradient method we have to solve the equation,

$$Bu = f_n \quad \text{for each} \quad n = 1, 2, \dots .$$

The method can be applied effectively if we can choose an operator B for which B^{-1} can be easily calculated.

STATEMENT 2. The gradient method can be applied to the equations of the form

$$Ax = f$$

where A may not be a symmetric operator. It can be done by taking $g = A^* f$ where A^* is the adjoint of the operator A. Then we are led to the new equation that has the following pleasant form:

$$A^* Ax = g . \tag{14}$$

Clearly $A^* A$ is a symmetric semidefinite operator. If $A^* A$ happens to be a strictly positive definite operator then the gradient method works as in the case of equation (3).

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ANALYTICITY OF THE SOLUTION OF BOUNDARY VALUE PROBLEMS FOR A SELF-ADJOINT ORDINARY DIFFERENTIAL EQUATION WITH POLYNOMIAL COEFFICIENTS VIA GRADIENT METHOD

By

A. SHAMANDY and A. EL-NENAE

Faculty of Sciences, Mansoura University, Egypt

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In this paper, using the gradient method, it is shown that the solution of a boundary value problem for a self-adjoint ordinary linear differential equation with polynomial coefficients is an analytic function. The proof is based on the fact that the gradient method provides a series of function with a very high speed convergence.

Let $I=[a, b]$ be a finite closed interval and

$$p_0, p_1, \ldots, p_N$$

polynomials satisfying the following conditions.

(i) For every $k=0, 1, \ldots, N-1$ $p_k(x) \geq 0$ $(x \in I)$

(ii) there exists a constant $m>0$ such that

$$p_N(x) \geq m \quad (x \in I).$$

Denote by $C_0^{(2N)}$ the set of such functions $u: I \to \mathbf{R}$ that are $2N$-times continuously differentiable in I and satisfy the homogeneous boundary conditions:

$$u^{(k)}(a)=u^{(k)}(b)=0 \quad (k=0, 1, \ldots, N-1). \tag{1}$$

Consider the self-adjoint linear differential operator of order $2N$ defined on the class of functions $C_0^{(2N)}$ by the equation

$$Au = \sum_{k=0}^{N} (-1)^k \frac{d^k}{dx^k}\left(P_k \frac{d^k u}{dx^k}\right).$$

For the special case $p_0=p_1=\ldots=p_{N-1}=0$, $p_N=1$ define

$$B=(-1)^N \frac{d^{2N}}{dx^{2N}}.$$

In [2] we have proved that besides the constant $m>0$ in condition (ii) there exists a constant M such that for every $u\in C_0^{(2N)}$ we have

$$m\,\|u\|_N^2\leq\langle Au,\,u\rangle\leq M\,\|u\|_N^2\,,$$

where $\langle\cdot,\,\cdot\rangle$ denotes the scalar product in $L_2(I)$ and

$$\|u\|_N^2=\langle Bu,\,u\rangle=\sum_{k=0}^{N}\|u^{(k)}\|_{L_2(I)}^2,\quad (u\in C_0^{(2N)})\,.$$

Let f be a given polynomial and consider the differential equation

$$Au=f \tag{2}$$

having a unique solution $u\in C_0^{(2N)}$ under conditions above.

THEOREM. *The solution u^* of the equation* (2) *satisfying the boundary conditions* (1) *is analytic in the closed interval I.*

PROOF. Apply the gradient method described in [1] and [2] to the approximate calculation of the solution u^*. Start from the initial function $u_0=0$. Then the method provides a series of functions (u_n) for which

$$u_n=u_{n-1}+t_n v_n\quad (n=1,2,\ldots)$$

where t_n is an appropriate constant while the solution v_n of the differential equation

$$Bv_n=f_n$$

belongs to the class of functions $C_0^{(2N)}$, where

$$f_n=Au_{n-1}-f\,.$$

For obtaining the first approximation u_1 we have to solve the differential equation

$$Bv_1=-f$$

which has a polynomial right-hand side. A simple calculations shows that the solution of this equation in the class $C_0^{(2N)}$ can be written in the form

$$v_1(x)=(b-x)^N(x-a)^N P(x)\quad (x\in I)$$

where P is a polynomial having the same degree as f. Hence, it follows that the first approximation

$$u_1=u_0+t_1 v_1=t_1 v_1$$

is also a polynomial having a degree $2N+\deg f$ where $\deg f$ stands for the degree of the polynomial f.

It can be easily proved by induction that every function u_n will be a polynomial for the degree of which the following estimation holds.

$$d_n=\deg u_n\leq(2N+d)\,n\quad (n=1,2,\ldots)$$

where d is the maximum of the degrees of polynomials $p_0,p_1,\ldots,p_N$ and f.

From the results of [1] and [2] we get the following estimation

$$\|u^*-u_n\|_N \leq Kq^n \quad (n = 1, 2, \ldots)$$

where K is a constant and $q = \dfrac{M-m}{M+m}$.

Introduce the notation $q_0 = q^{1/2N+d}$ and take into account that for every $u \in C_0^{(2N)}$ the inequality

$$\max_I |u| \leq \|u\|_N$$

holds (see [2]). Then we get the inequality

$$\max_I |u^*-u_n| \leq Kq_0^{d_n} \quad (n = 1, 2, \ldots)$$

where $0 \leq q_0 < 1$ and u_n is a polynomial of degree d_n. Hence, by the well-known Berstein theorem, it follows that u^* is an analytic function in the interval.

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ESTIMATES ON THE EXISTENCE REGION OF PERIODIC SOLUTIONS OF PERTURBED VAN DER POL'S EQUATION

By

TRAN VAN NHUNG

Department of Mathematics, Budapest University of Technology

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The aim of this paper is to apply the estimates of M. FARKAS in [1] on the existence regions of periodic solutions of the n dimensional perturbed differential systems to perturbed van der Pol's equation. In the first section we repeat shortly the results of M. FARKAS for the general case. Applying these results in the second section we get explicit bounds for the parameters among which the existence of the periodic solutions of perturbed van der Pol's equation is ensured.

1. M. Farkas' estimates

In [1] the perturbed system of differential equations

(1.1) $$\dot{x} = f(x) + \mu g\left(\frac{t}{\tau}, x\right)$$

was considered, where dot means differentiation with respect to $t \in R$, $\mu \in R$ is a "small parameter", $\tau > 0$ is a real parameter, $x \in \Omega$, Ω is an open and bounded ball in R^n with center in the origin (or, in general, Ω is some open, connected and bounded region in R^n), the functions $f: \bar{\Omega} \to R^n$, $g: R \times \bar{\Omega} \to R^n$ belong to the C^2 class in the closure $\bar{\Omega}$ of the ball Ω and g is periodic in the variable t with period τ, i.e.

$$g(s+1, x) \equiv g(s, x), \quad s \in R, \quad x \in \bar{\Omega}.$$

It was also assumed that the unperturbed system

(1.2) $$\dot{x} = f(x)$$

has a non-constant periodic solution $p: R \to \Omega$ with period $\tau_0 > 0$ and the number 1 is a simple characteristic multiplier of the variational system of (1.1) corresponding to the periodic solution $p(t)$

(1.3) $$\dot{y} = f'_x(p(t))\, y.$$

In order to formulate M. Farkas' theorem we need the following notations and constants. Let us denote $p^0 = p(0) = \text{col}(p_1^0, \ldots, p_n^0)$ and $\dot{p}^0 = \dot{p}(0) = \text{col}(\dot{p}_1^0, \ldots, \dot{p}_n^0)$. It is assumed, without loss of generality, that $\dot{p}_1^0 \neq 0$, $\dot{p}_i^0 = 0$ $(i = 2, \ldots, n)$. The norm of vectors and matrices is defined as follows:

$$\nu = \text{col}(\nu_1, \ldots, \nu_m), \quad \|\nu\| = \max_i |\nu_i|,$$

$$M = [m_{ik}] \quad (i = 1, \ldots, n;\ k = 1, \ldots, m), \quad \|M\| = m \max_{i,k} |m_{ik}|,$$

$$E = [e_{ikl}] \quad (i, k, l = 1, \ldots, n), \quad \|E\| = n^2 \max_{i,k,l} |e_{ikl}|.$$

Put

$$\sigma = \text{dist}(\Gamma, \partial\Omega) > 0, \tag{1.4}$$

where Γ denotes the path of the periodic solution $p(t)$ and $\partial\Omega$ is the boundary of ball Ω;

$$F_0 = \max_{x \in \bar{\Omega}} \|f(x)\|, \quad G_0 = \max_{\substack{x \in \bar{\Omega} \\ s \in R}} \|g(s, x)\|,$$

$$F_1 = \max_{x \in \Omega} \|f'_x(x)\|, \quad G_1 = \max_{\substack{x \in \bar{\Omega} \\ s \in R}} \|g'_x(s, x)\|, \tag{1.5}$$

$$F_2 = \max_{x \in \bar{\Omega}} \|f''_{xx}(x)\|, \quad G_s = \max_{\substack{x \in \Omega \\ s \in R}} \|g'_s(s, x)\|;$$

$$J = \begin{bmatrix} \dot{p}_1^0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} + Y(\tau_0) - I, \tag{1.6}$$

where $Y(t)$ is the fundamental matrix solution of the variational system (1.3) which assumes the unit matrix at $t = 0$, i.e. $Y(0) = I$;

$$K = \max_{-\frac{\tau_0}{2} \leq t \leq \tau_0} \|Y(t)\|, \quad K_{-1} = \max_{-\frac{\tau_0}{2} \leq t \leq \tau_0} \|Y^{-1}(t)\|; \tag{1.7}$$

$$P = \max_{0 \leq t \leq \tau_0} \|\dot{p}(t)\| \leq \frac{K}{n} \cdot |\dot{p}_1^0|. \tag{1.8}$$

Let α_i, β_i $(i = 1, 2, 3)$ be arbitrary positive numbers satisfying the following inequalities, respectively:

$$\frac{3}{2} G_0 \tau_0 \alpha_1 + \beta_1 < \sigma \exp\left(-\frac{3}{2} F_1 \tau_0\right), \tag{1.9}$$

$$a_2\left\{\frac{3}{2}F_1 G_0 \tau_0 \exp\left(\frac{3}{2}F_1 \tau_0\right)+G_0+4G_s \exp\left[\frac{3}{2}\tau_0 (F_1+G_1 a_2)\right]\right\}+$$

$$+\beta_2 (1+F_0) F_1 \exp\left(\frac{3}{2}F_1 \tau_0\right)<\frac{1}{n\|J^{-1}\|}, \tag{1.10}$$

$$\left\{\frac{3}{2}a_3\left[\frac{3}{2}F_2 G_0 \tau_0^2 \exp\left(\frac{3}{2}F_1 \tau_0\right)+G_1 \tau_0\right]+\right.$$

$$\left.+\beta_3\left[\frac{3}{2}(1+F_0) F_2 \tau_0 \exp\left(\frac{3}{2}F_1 \tau_0\right)+F_1\right]\right\}\exp\left(\frac{3}{2}\tau_0 G_1 a_3\right)<$$

$$<\frac{1}{\|J^{-1}\| K}\exp\left(-\frac{3}{2}F_1 \tau_0\right). \tag{1.11}$$

Further, let us define

$$\alpha = \min\left(a_1, a_2, a_3, \frac{\tau_0}{2}\right),$$

$$\beta = \min\left(\beta_1, \beta_2, \beta_3, \frac{\tau_0}{2}\right); \tag{1.12}$$

$$H_1 = n\left\{\alpha\left(\frac{3}{2}F_1 G_0 \tau_0 \exp\left(\frac{3}{2}F_1 \tau_0\right)+G_0+4G_s \exp\left[\frac{3}{2}\tau_0 (F_1+G_1 \alpha)\right]\right)+\right.$$

$$\left.+\beta (1+F_0) F_1 \exp\left(\frac{3}{2}F_1 \tau_0\right)\right\},$$

$$H_2 = \left\{\frac{3}{2}\alpha\left(\frac{3}{2}F_2 G_0 \tau_0^2 \exp\left(\frac{3}{2}F_1 \tau_0\right)+G_1 \tau_0\right)+\right.$$

$$\left.+\beta\left(\frac{3}{2}(1+F_0) F_2 \tau_0 \exp\left(\frac{3}{2}F_1 \tau_0\right)+F_1\right)\right\} K \exp\left[\frac{3}{2}\tau_0 (F_1+G_1 \alpha)\right],$$

$$H = \max(H_1, H_2); \tag{1.13}$$

$$A_1 = \|J^{-1}\| K K_{-1} G_0 \tau_0 + \frac{\|J^{-1}\|}{1-\|J^{-1}\| H}\left\{\left[\frac{3}{2}G_s (\alpha+\beta)+\right.\right.$$

$$+\frac{3}{2}(G_1 \tau_0 + F_2 K K_{-1} G_0 \tau_0^2)\left(\frac{3}{2}G_0 \tau_0 \alpha+\beta+F_0 \beta\right)\exp\left(\frac{3}{2}F_1 \tau_0\right)+$$

$$\left.+\frac{3}{2}G_1 K K_{-1} G_0 \tau_0^2 \alpha + K K_{-1} G_0 \beta\right]\exp\left(\frac{3}{2}\tau_0 (F_1+G_1 \alpha)\right)+$$

$$\left.+\|J^{-1}\| H K K_{-1} G_0 \tau_0\right\}, \tag{1.14}$$

$$\Delta_2 = \frac{\|J^{-1}\|}{1-\|J^{-1}\| H}\left\{F_1\left(\frac{3}{2}G_0\tau_0\alpha+\beta+F_0\beta\right)\exp\left(\frac{3}{2}F_1\tau_0\right)+\right.$$

$$+G_0\alpha+\left[\frac{3}{2}\left(\frac{3}{2}G_0\tau_0\alpha+\beta+F_0\beta\right)F_2\tau_0\exp\left(\frac{3}{2}F_1\tau_0\right)+\right.$$

$$\text{(1.15)}\qquad \left.\left.+\frac{3}{2}G_1\tau_0\alpha+F_1\beta\right]P\exp\left(\frac{3}{2}\tau_0(F_1+G_1\alpha)\right)\right\},$$

$$\text{(1.16)}\qquad \Delta = 2\max(\Delta_1, \Delta_2).$$

Finally we introduce the notation

$$\text{(1.17)}\qquad U = \left\{(\mu,\varphi)\in R^2 : |\mu| < \min\left(\alpha, \frac{\beta}{\Delta}\right),\ \ |\varphi| < \min\left(\alpha, \frac{\beta}{\Delta}\right)\right\}.$$

THEOREM 1. (M. Farkas, [1]). *For each $(\mu,\varphi)\in U$ there exist an unique $\tau(\mu,\varphi)\in R$ and an unique $h(\mu,\varphi) = \mathrm{col}\,(0, h_2(\mu,\varphi), \ldots, h_n(\mu,\varphi))\in R^n$ such that the solution of system*

$$\dot{x} = f(x)+\mu g\left(\frac{t}{\tau(\mu,\varphi)}, x\right)$$

which at $t=\varphi$ assumes the value $x = p^0+h(\mu,\varphi)$ is periodic with period $\tau(\mu,\varphi)$, the functions $\tau: U\to R$, $h: U\to R^n$ have the properties $\tau\in C^1_U$, $h\in C^1_U$, $|\tau(\mu,\varphi)-\tau_0|<\beta$, $\|h(\mu,\varphi)\|<\beta$ in U, $\tau(0,0)=\tau_0$, $h(0,0)=0$.

2. Application to perturbed van der Pol's equation

The unperturbed van der Pol's differential equation

$$\text{(2.1)}\qquad \ddot{u}+m(u^2-1)\dot{u}+u = 0,$$

where m is a positive constant, is known (see N. LEVINSON [3]) to have a non-constant periodic solution with period depending on m. This solution will be denoted by $u_0(t)$ and its (least positive) period by τ_0. It is assumed, without loss of generality ([4]), that

$$\text{(2.2)}\qquad u_0(0) = a > 0,\quad \dot{u}_0(0) = 0.$$

Let us introduce the notations:

$$\text{(2.3)}\qquad U_0 = \max_{0\le t\le\tau_0}|u_0(t)|,\quad \dot{U}_0 = \max_{0\le t\le\tau_0}|\dot{u}_0(t)|.$$

It is also well known (see M. URABE [5], pp. 216–218) that the path of the τ_0-periodic solution $(u_0(t), \dot{u}_0(t))$ of the system $\dot{u} = v$, $\dot{v} = -m(u^2-1)v-u$, i.e. the closed orbit $G = \{(u,v)\in R^2 : u = u_0(t),\ v = \dot{u}_0(t),\ t\in[0,\tau_0]\}$, encloses the origin in the phase plane (u, v) and it crosses the v-axis at two symmetric points $(0, b)$ and $(0, -b)$ $(b>0)$.

By using LOUD's substitution ([4])

$$x_1 = -\dot{u} + m\left(\frac{1}{3}a^3 - a\right) - m\left(\frac{1}{3}u^3 - u\right),$$

(2.4)

$$x_2 = u - a\,,$$

from the scalar equation (2.1) we get the equivalent two dimensional system

(2.5) $$\dot{x} = f(x)\,,$$

where $x = (x_1, x_2)$ and $f(x)$ has the components

$$f_1(x) = x_2 + a\,,$$

(2.6)

$$f_2(x) = -x_1 + m\left(\frac{1}{3}a^3 - a\right) - m\left[\frac{1}{3}(x_2 + a)^3 - (x_2 + a)\right].$$

Corresponding to the τ_0-periodic solution $u_0(t)$ of (2.1) the system (2.5) possesses the τ_0-periodic solution $p(t) = (p_1(t), p_2(t))$, where

$$p_1(t) = -\dot{u}_0(t) + m\left[\frac{1}{3}a^3 - a\right] - m\left[\frac{1}{3}u_0^3(t) - u_0(t)\right],$$

(2.7)

$$p_2(t) = u_0(t) - a.$$

It follows from (2.7) and (2.2) that $\dot{p}_1(t) = u_0(t)$, $\dot{p}_2(t) = \dot{u}_0(t)$ and

(2.8) $$p^0 = p(0) = (p_1^0, p_2^0) = (0, 0)\,,$$

(2.9) $$\dot{p}^0 = \dot{p}(0) = (\dot{p}_1^0, \dot{p}_2^0) = (a, 0)\,.$$

Now we consider the perturbed van der Pol's equation of the form

(2.10) $$\ddot{u} + m(u^2 - 1)\dot{u} + u = \mu\gamma\left(\frac{t}{\tau}, u, \dot{u}\right),$$

where $t \in R$, $\mu \in R$ is a small parameter, $|\mu| < \mu_0$ for some $\mu_0 > 0$, $\tau > 0$ is a real parameter, $|\tau - \tau_0| < \tau_1$ for some $0 < \tau_1 < \tau_0$, $(u, \dot{u}) \in \Omega$,

(2.11) $$\Omega = \{(u, v) \in R^2 : u^2 + v^2 < r^2\}$$

with some finite number $r > 0$ such that the closed orbit G is contained in Ω and the function $\gamma : R \times \overline{\Omega} \to R$ belongs to the C^2 class and γ is periodic in t with period τ. The equivalent system obtained from equation (2.10) by (2.4) can be written in the form

(2.12) $$\dot{x} = f(x) + \mu g\left(\frac{t}{\tau}, x\right),$$

where x and f are the same as in (2.5) and $g\left(\frac{t}{\tau}, x\right)$ has the components

$$g_1\left(\frac{t}{\tau}, x\right) = -\gamma\left(\frac{t}{\tau}, x_2+a, -x_1+m\left(\frac{1}{3}a^3-a\right)-m\left[\frac{1}{3}(x_2+a)^3-(x_2+a)\right]\right),$$

$$g_2\left(\frac{t}{\tau}, x\right) = 0.$$

(2.13)

Now vector x in system (2.12) varies in the image S of ball Ω by the transformation (2.4). The set

$$S = \left\{(x_1, x_2) \in R^2 : (x_2+a)^2 + \left[-x_1+m\left(\frac{1}{3}a^3-a\right)-m\left(\frac{1}{3}(x_2+a)^3-(x_2+a)\right)\right]^2 < r^2\right\} \tag{2.14}$$

is evidently an open, connected and bounded domain.

In order to apply M. Farkas' theorem to system (2.12) now we must determine the quantities σ, F_0, F_1, F_2, G_0, G_s, G_1, $\|J^{-1}\|$, K, K_{-1} and P for this case according to the general formulae (1.4), (1.5), (1.6), (1.7) and (1.8).

In this case we have

$$\sigma = \text{dist}(\Gamma, \partial S) > 0, \tag{2.15}$$

where Γ denotes the path of the periodic solution $p(t)$ (see (2.7)) in the plane (x_1, x_2), i.e. Γ is the image of G by (2.4),

$$F_0 = \max_{x\in\bar{S}} \left\|\left\{x_2+a, -x_1+m\left(\frac{1}{3}a^3-a\right)-m\left[\frac{1}{3}(x_2+a)^3-(x_2+a)\right]\right\}\right\| = r,$$

(2.16)

$$F_1 = \max_{x\in\bar{S}} \left\|\begin{pmatrix} 0 & 1 \\ -1 & -m[(x_2+a)^2-1]\end{pmatrix}\right\| = 2\max_{x\in\bar{S}} \max\left(1, m|(x_2+a)^2-1|\right) =$$

$$= 2\max[1, m\max(1, r^2-1)],$$

(2.17)

and

$$F_2 = 8mr, \tag{2.18}$$

because

$$f''_{1x_1x_1}(x) = f''_{1x_1x_2}(x) = f''_{1x_2x_1}(x) = f''_{1x_2x_2}(x) =$$
$$= f''_{2x_1x_1}(x) = f''_{2x_1x_2}(x) = f''_{2x_2x_1}(x) = 0$$

and
$$f''_{2x_2x_2}(x) = -2m(x_2+a).$$

Let the following notations be introduced:

$$\max_{\substack{(u,\dot u)\in\bar\Omega\\ s\in R}} |\gamma(s, u, \dot u)| = \gamma_0, \qquad \max_{\substack{(u,\dot u)\in\bar\Omega\\ s\in R}} |\gamma'_s(s, u, \dot u)| = \gamma_s,$$

$$\max_{\substack{(u,\dot u)\in\bar\Omega\\ s\in R}} |\gamma'_u(s, u, \dot u)| = \gamma_1, \qquad \max_{\substack{(u,\dot u)\in\bar\Omega\\ s\in R}} |\gamma'_{\dot u}(s, u, \dot u)| = \gamma_2. \tag{2.19}$$

It is clear that

$$G_0 = \max_{\substack{x\in\bar{S}\\ s\in R}} \left|\gamma\left(s, x_2+a, -x_1+m\left(\frac{1}{3}a^3-a\right)-m\left[\frac{1}{3}(x_2+a)^3-(x_2+a)\right]\right)\right| = \tag{2.20}$$
$$= \gamma_0 ,$$

$$G_s = \max_{\substack{x\in\bar{S}\\ s\in R}} \left|\gamma_s'\left(s, x_2+a, -x_1+m\left(\frac{1}{3}a^3-a\right)-m\left[\frac{1}{3}(x_2+a)^3-(x_2+a)\right]\right)\right| = \tag{2.21}$$
$$= \gamma_s .$$

The matrix $g_x'(s, x)$ has the form

$$g_x'(s, x) = \begin{bmatrix} \gamma_u' & -\gamma_u'+m\left[(x_2+a)^2-1\right]\gamma_u' \\ 0 & 0 \end{bmatrix},$$

hence we have

$$\|g_x'(s, x)\| \le 2\max\left(|\gamma_u'|, |\gamma_u'|+m|\gamma_u'|\,|(x_2+a)^2-1|\right) \le$$
$$\le 2\max\left[\gamma_2, \gamma_1+m\gamma_2\max(1, r^2-1)\right],$$

and from this it follows that

$$G_1 \le G_1' \overset{\text{def}}{=} 2\max\left[\gamma_2, \gamma_1+m\gamma_2\max(1, r^2-1)\right]. \tag{2.22}$$

Now we find the Jacobi matrix J and after that we shall calculate the norm of its inverse matrix J^{-1}. In this case according to formula (1.6) the matrix J is written in the form

$$J = \begin{bmatrix} \dot{p}_1^0 & 0 \\ 0 & 0 \end{bmatrix} + Y(\tau_0) - I , \tag{2.23}$$

where $\dot{p}_1^0 = a$ (see (2.9)), $Y(t)$ is the fundamental matrix solution of the variational system

$$\dot{y} = \begin{bmatrix} 0 & 1 \\ -1 & -m\left(u_0^2(t)-1\right) \end{bmatrix} y \tag{2.24}$$

with $Y(0) = I = \operatorname{diag}(1, 1)$. One solution of system (2.24) is clearly $(u_0(t), \dot{u}_0(t))$, which is τ_0-periodic. The system (2.24) has the second solution $(v_0(t), \dot{v}_0(t))$, where

$$v_0(t) = u_0(t) \int_0 u_0^{-2}(s) \exp\left(-m\int_0^s \left[u_0^2(\alpha)-1\right] d\alpha\right) ds , \tag{2.25}$$

for which

$$v_0(0) = 0, \quad \dot{v}_0(0) = a^{-1} \tag{2.26}$$

(see W. S. LOUD [4]). It is clear from (2.2) and (2.26) that two solutions $(u_0(t), \dot{u}_0(t))$, $(v_0(t), \dot{v}_0(t))$ of system (2.24) are linearly independent, hence the fundamental matrix solution $Y(t)$ $(Y(0) = I)$ of system (2.24) has the form

$$Y(t) = \begin{bmatrix} a^{-1} u_0(t) & a v_0(t) \\ a^{-1} \dot{u}_0(t) & a \dot{v}_0(t) \end{bmatrix}. \tag{2.27}$$

By Liouville's formula we have

$$W(t) = \det Y(t) = \exp\left(-m \int_0^t [u_0^2(\alpha) - 1]\, d\alpha\right).$$

Let us denote the two characteristic multipliers of system (2.24) by ϱ_1 and ϱ_2, then as it is well known that $\varrho_1 = 1$ and

$$\varrho_2 = W(\tau_0) = \det Y(\tau_0) = \exp\left(-m \int_0^{\tau_0} [u_0^2(t) - 1]\, dt\right) > 0. \tag{2.28}$$

Furthermore we note that

$$\varrho_2 < 1 \tag{2.29}$$

for arbitrary positive values of the parameter m (see M. URABE [5]). By (2.28) the inequality (2.29) is equivalent to the inequality

$$\int_0^{\tau_0} [u_0^2(t) - 1]\, dt > 0. \tag{2.30}$$

It follows from (2.27) and (2.2) that

$$Y(\tau_0) = \begin{bmatrix} 1 & a v_0(\tau_0) \\ 0 & a \dot{v}_0(\tau_0) \end{bmatrix},$$

or

$$Y(\tau_0) = \begin{bmatrix} 1 & a v_0(\tau_0) \\ 0 & \varrho_2 \end{bmatrix} \tag{2.31}$$

because $a\dot{v}_0(\tau_0) = \det Y(\tau_0) = \varrho_2$. Substituting $\dot{p}_1^0 = a$ and the last expression into (2.23) we get

$$J = \begin{bmatrix} a & a v_0(\tau_0) \\ 0 & \varrho_2 - 1 \end{bmatrix},$$

and its inverse matrix is

$$J^{-1} = \begin{bmatrix} a^{-1} & v_0(\tau_0)(1-\varrho_2)^{-1} \\ 0 & -(1-\varrho_2)^{-1} \end{bmatrix}.$$

Therefore we have

$$\|J^{-1}\| = 2 \max [a^{-1},\ (1-\varrho_2)^{-1},\ v_0(\tau_0)(1-\varrho_2)^{-1}], \tag{2.32}$$

where by (2.25)

$$v_0(\tau_0) = a \int_0^{\tau_0} u_0^{-2}(s) \exp\left(-m \int_0^s [u_0^2(\alpha) - 1]\, d\alpha\right) ds > 0 .$$

Let us denote

$$V_0 = \max_{-\frac{\tau_0}{2} \leq t \leq \tau_0} |v_0(t)|, \quad \dot{V}_0 = \max_{-\frac{\tau_0}{2} \leq t \leq \tau_0} |\dot{v}_0(t)|,$$

(2.33)

$$W_0 = \min_{-\frac{\tau_0}{2} \leq t \leq \tau_0} W(t) > 0 .$$

The inverse matrix of the fundamental matrix solution $Y(t)$ has the form

$$Y^{-1}(t) = \begin{bmatrix} a\, \dot{v}_0(t)\, W^{-1}(t) & -a\, v_0(t)\, W^{-1}(t) \\ -a^{-1}\, \dot{u}_0(t)\, W^{-1}(t) & a^{-1}\, u_0(t)\, W^{-1}(t) \end{bmatrix}.$$

It is easy to see that the following estimates for K and K_{-1} hold:

(2.34) $$K \leq K' \leq K'' ,$$

where

(2.35) $$K' = 2 \max(a^{-1} U_0,\ a^{-1} \dot{U}_0,\ aV_0,\ a\dot{V}_0),$$

(2.36) $$K'' = 2 \max(a^{-1} r,\ aV_0,\ a\dot{V}_0),$$

and

(2.37) $$K_{-1} \leq K'_{-1} \leq K''_{-1},$$

where

(2.38) $$K'_{-1} = 2W_0^{-1} \max(a^{-1} U_0,\ a^{-1} \dot{U}_0,\ aV_0,\ a\dot{V}_0),$$

(2.39) $$K''_{-1} = 2W_0^{-1} \max(a^{-1} r,\ aV_0,\ a\dot{V}_0).$$

From (1.8), (2.9) and (2.34) we obtain

(2.40) $$P \leq P' \leq P'' ,$$

where

(2.41) $$P' = \frac{1}{2} aK'$$

and

(2.42) $$P'' = \frac{1}{2} aK'' .$$

Denote

$$(2.43)\qquad U' = \left\{(\mu,\varphi)\in R^2 : |\mu| < \min\left(\mu_0, \alpha, \frac{\beta}{\Delta}\right),\ |\varphi| < \min\left(\alpha, \frac{\beta}{\Delta}\right)\right\},$$

where α, β, Δ are determined by (1.9), (1.10), (1.11), (1.12) (1.13), (1.14), (1.15), (1,16) in which σ, F_0, F_1, F_2, G_0, G_s, $\|J^{-1}\|$ are given respectively by (2.15), (2.16), (2.17) (2.18), (2.20), (2.21), (2.32) and G_1, K, K_{-1}, P are replaced respectively by their upper bounds G_1', K' (or K''), K'_{-1} (or K''_{-1}), P' (or P'') in (2.22), (2.35), (2.36), (2.38), (2.39), (2.41), (2.42).

Now we have all necessary notations and constants to formulate the following result.

THEOREM 2. *If the inequality* (2.30) *is satisfied then there exist two C^1-maps $\tau, h: U'\to R$ such that for each $(\mu,\varphi)\in U'$ the solution*

$$u\left(t;\varphi,\ a+h(\mu,\varphi),\ mh(\mu,\varphi)\left(1-a^2-ah(\mu,\varphi)-\frac{1}{3}h^2(\mu,\varphi)\right),\mu,\tau(\mu,\varphi)\right)$$

of the perturbed van der Pol's equation

$$(2.10')\qquad \ddot{u}+m(u^2-1)\dot{u}+u=\mu\gamma\left(\frac{t}{\tau(\mu,\varphi)},u,\dot{u}\right)$$

which at $t=\varphi$ assumes the value

$$(u,\dot{u})=\left(a+h(\mu,\varphi),\ mh(\mu,\varphi)\left(1-a^2-ah(\mu,\varphi)-\frac{1}{3}h^2(\mu,\varphi)\right)\right)$$

is periodic with period $\tau(\mu,\varphi)$ and the maps τ, h have the properties

$$|\tau(\mu,\varphi)-\tau_0|<\min(\tau_1,\beta),\quad |h(\mu,\varphi)|<\beta\quad in\quad U',\quad \tau(0,0)=\tau_0,\quad h(0,0)=0.$$

PROOF. Theorem 2 is proved by applying Theorem 1 of M. Farkas to the two dimensional system (2.12). In this case by assumption (2.30) we note that the number 1 is a simple characteristic multiplier of system (2.24). According to Theorem 1 for system (2.12) there exist two C^1-maps $\tau, h: U\to R$ such that the solution

$$\Big(x_1\big(t;\varphi,(0,h(\mu,\varphi)),\ \mu,\tau(\mu,\varphi)\big),\ x_2\big(t;\varphi,(0,h(\mu,\varphi)),\ \mu,\tau(\mu,\varphi)\big)\Big)$$

of system (2.12) which at $t=\varphi$ assumes the value $(0,h(\mu,\varphi))$ is $\tau(\mu,\varphi)$-periodic and the two maps τ, h possess the properties expressed in Theorem 1. Then by (2.4) the function $u(t,\mu,\varphi)=x_2\big(t;\varphi,(0,h(\mu,\varphi)),\mu,\tau(\mu,\varphi)\big)+a$ is the $\tau(\mu,\varphi)$-periodic solution of equation (2.10′) which satisfies the initial conditions expressed in the Theorem.

It is easy to see that domain U' in (2.43) is contained in the domain U of the form (1.17) for system (2.12). The proof of Theorem 2 is completed by using the restrictions of the maps τ and h on the region U'.

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ÜBER LÖSUNGEN ASYMPTOTISCH PERIODISCHER STOCHASTISCHER SYSTEME

Von

TRAN VAN NHUNG

Lehrstuhl für Mathematik, Technische Universität, Budapest

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In dieser Arbeit betrachten wir einige asymptotische Eigenschaften von Lösungen asymptotisch periodischer stochastischer Systeme auf Grund einiger Resultate von H. BUNKE in [1], [2].

Im weiteren brauchen wir folgende einfache Begriffe. Einen stochastischen Prozeß $\xi_t (t \in T)$, dessen Realisierungen fast alle auf T stetig sind, nennen wir R-stetig auf T ([1], S. 12). Ein stochastischer Prozeß $\xi_t (t \in T)$ heißt streng Θ-periodisch, wenn alle endlichdimensionalen Verteilungsfunktionen der Gestalt

$$F(x_1, \ldots, x_k;\ t_1+\tau, \ldots, t_k+\tau) \quad (t_1, \ldots, t_k \in T)$$

Θ-periodische Funktionen in τ mit der gleichen Periode Θ sind. Ein stochastischer Prozeß $\xi_t (t \in T)$ heißt streng stationär, wenn alle endlichdimensionalen Verteilungsfunktionen der Gestalt $F(x_1, \ldots, x_k;\ t_1+\tau, \ldots, t_k+\tau)$ unabhängig von τ sind ([1], S. 38).

1. In [1], S. 83, hat H. BUNKE das folgende lineare periodische stochastische System betrachtet:

$$\frac{dx_t}{dt} = A_t x_t + z_t, \tag{1.1}$$

wobei A_t ein auf R^1 R-stetiger streng Θ-periodischer stochastischer $n \times n$-Matrixprozeß und z_t ein von A_t unabhängiger auf R^1 R-stetiger streng Θ-periodischer stochastischer n-Vektorprozeß zweiter Ordnung mit

$$\sup_{t \in [t_0, t_0+\Theta]} E\, \|z_t\|^2 < k \quad (k = \text{const}) \tag{1.2}$$

ist. Mit $\|\cdot\|$ bezeichnen wir die Euclidische Norm von Vektoren bzw. von Matrizen. $E(\xi)$ ist der Erwartungswert von ξ. $(\Omega, \mathcal{U}, P)$ bezeichnet den Wahrscheinlichkeitsraum.

DEFINITION ([1], S. 83). Wir nennen die Lösung x_t von (1.1) asymptotisch Θ-periodisch, falls

$$\lim_{t\to\infty} [E x_{t+\Theta} - E x_t] = 0 \tag{1.3}$$

und

$$\lim_{t,s\to\infty} [K^x(t+\Theta, s+\Theta) - K^x(t,s)] = 0\,, \tag{1.4}$$

wobei $K^x(t,s) = E\{(x_t - E x_t)(x_s - E x_s)^T\}$ die Kovarianzfunktion des Prozeßes x_t ist, gilt.

H. BUNKE hat die folgende hinreichende Bedingung dafür angegeben, daß jede Lösung von (1.1) asymptotisch Θ-periodisch ist.

SATZ 1.1 ([1], S. 83). *Gilt für jede Lösung y_t von*

$$\frac{dy_t}{dt} = A_t y_t \tag{1.5}$$

zu jeder Anfangsbedingung $(y_0, \tau) \in L^2(\Omega, \mathcal{U}, P) \times R^1$

$$E\,\|y_t\|^2 \le \alpha\, E\,\|y_0\|^2\, e^{-\varrho(t-\tau)} \tag{1.6}$$

mit positiven Konstanten α und ϱ, dann ist jede Lösung x_t von (1.1) *zu einer Anfangsbedingung $(x_0, t_0) \in L^2(\Omega, \mathcal{U}, P) \times R^1$, wobei x_0 vom Prozeß (A_t, z_t) unabhängig ist, asymptotisch Θ-periodisch.*

Nun betrachten wir die Gleichung

$$\frac{d\xi_t}{dt} = A_t \xi_t + z_t + \zeta_t\,, \tag{1.7}$$

wobei A_t und z_t wie bei (1.1) sind und ζ_t ein von A_t unabhängiger auf R^1 R-stetiger stochastischer n-Vektorprozeß zweiter Ordnung mit $E\|\zeta_t\|^2 \in C(R^1)$ und

$$\sup_{t\in R^1} E\,\|\zeta_t\|^2 < \infty$$

ist.

SATZ 1.2. *Es seien* (1.6) *und die Bedingung*

$$\lim_{t\to\infty} E\,\|\zeta_t\|^2 = 0 \tag{1.8}$$

erfüllt, dann ist jede Lösung ξ_t von (1.7) *zu einer Anfangsbedingung $(\xi_0, t_0) \in$ $\in L^2(\Omega, \mathcal{U}, P) \times R^1$*, wobei ξ_0 *vom Prozeß (A_t, z_t) unabhängig ist, asymptotisch Θ-periodisch.*

BEWEIS. ξ_t sei eine beliebige Lösung von (1.7) zu einer Anfangsbedingung $(\xi_0, t_0) \in L^2(\Omega, \mathcal{U}, P) \times R^1$, wobei ξ_0 vom Prozeß (A_t, z_t) unabhängig ist. x_t sei die Lösung von (1.1) zu der Anfangsbedingung (ξ_0, t_0) und η_t^0 die Lösung von

$$\frac{d\,\eta_t}{dt} = A_t \eta_t + \zeta_t \tag{1.9}$$

zu der Anfangsbedingung $(0, t_0)$, dann gilt

$$\xi_t = x_t + \eta_t^0 . \tag{1.10}$$

Nach dem Satz 1.1 ist x_t asymptotisch Θ-periodisch, d. h. (1.3) und (1.4) erfüllt sind. Im folgenden zeigen wir, daß für jede Lösung η_t von (1.9) zu jeder Anfangsbedingung $(\eta_0, t_0) \in L^2(\Omega, \mathcal{U}, P) \times R^1$

$$\lim_{t\to\infty} E\,\|\eta_t\|^2 = 0 \tag{1.11}$$

gilt. Q_t sei die Fundamentalmatrix von (1.5) mit $Q_{t_0} = I$ ($n \times n$-Einheitsmatrix), dann gibt es wegen (1.6) Konstanten β und ϱ, so daß

$$E\,\|Q_t Q_s^{-1}\|^2 \le \beta\, e^{-\varrho(t-s)} \quad (t \ge s) \tag{1.12}$$

gilt. Aus der Gleichung

$$\eta_t = Q_t \eta_0 + \int_{t_0} Q_t Q_s^{-1} \zeta_s\, ds$$

erhalten wir mit (1.6), (1.12) und nach Anwendung der Schwarzschen Ungleichung folgende Abschätzungen

$$E\,\|\eta_t\|^2 \le 2E\,\|Q_t\eta_0\|^2 + 2E\left\|\int_{t_0}^t Q_t Q_s^{-1}\zeta_s\, ds\right\|^2 \le$$

$$\le 2E\,\|Q_t\eta_0\|^2 + 2\int_{t_0}^t\int_{t_0}^t E\,(\|Q_t Q_s^{-1}\|\,\|Q_t Q_u^{-1}\|)\,E\,(\|\zeta_s\|\,\|\zeta_u\|)\,ds\,du \le$$

$$\le 2E\,\|Q_t\eta_0\|^2 + 2\left[\int_{t_0}^t (E\,\|Q_t Q_s^{-1}\|^2)^{1/2}(E\,\|\zeta_s\|^2)^{1/2}\,ds\right]^2 \le$$

$$\le 2\alpha\, E\,\|\eta_0\|^2 e^{-\varrho(t-t_0)} + 2\beta\, e^{-\varrho t}\left[\int_{t_0}^t e^{(\varrho/2)s}(E\,\|\zeta_s\|^2)^{1/2}\,ds\right]^2 .$$

Hieraus ist (1.11) nach der L'Hospital-Regel und mit (1.8) gezeigt.

Wegen (1.10), (1.3) und (1.11) gilt $\|E\,\xi_{t+\Theta} - E\,\xi_t\| \to 0$ $(t\to\infty)$. Aus (1.10) haben wir

$$K^\xi(t,s) = K^x(t,s) + K^{x\eta^0}(t,s) + K^{\eta^0 x}(t,s) + K^{\eta^0}(t,s),$$

wobei

$$K^{x\eta^0}(t,s) = E\,(x_t\,\eta_s^{0T}) - Ex_t\,(E\,\eta_s^0)^T$$

die gemeinsame Kovarianzfunktion von x_t und η_t^0 ist. Wir bemerken, daß für die Lösung x_t von (1.1)

$$E\,\|x_t\|^2 \le c \quad (c = \text{const}), \quad t \in R^1, \tag{1.13}$$

gilt ([1], S. 84). Aus (1.11), (1.13) folgen nach der Schwarzschen Ungleichung

$$\lim_{t,s\to\infty} \|K^{x\eta^0}(t+\Theta, s+\Theta) - K^{x\eta^0}(t,s)\| = 0\,,$$

$$\lim_{t,s\to\infty} \|K^{\eta^0 x}(t+\Theta, s+\Theta) - K^{\eta^0 x}(t,s)\| = 0\,,$$

$$\lim_{t,s\to\infty} \|K^{\eta^0}(t+\Theta, s+\Theta) - K^{\eta^0}(t,s)\| = 0\,.$$

Damit und mit (1.4) erhalten wir

$$\lim_{t,s\to\infty} \|K^{\xi}(t+\Theta, s+\Theta) - K^{\xi}(t,s)\| = 0\,.$$

Der Satz 1.2 ist bewiesen.

Im Fall der Gleichung

$$\frac{dy_t}{dt} = [A(t) + C_t]\, y_t\,, \tag{1.14}$$

wobei $A(t)$ eine auf R^1 stetige Θ-periodische deterministische $n\times n$-Matrixfunktion und C_t ein auf R^1 R-stetiger streng Θ-periodischer stochastischer $n\times n$-Matrixprozeß ist, zeigen wir eine hinreichende Bedingung für (1.6), die auf Koeffizienten von (1.14) gegeben ist.

Wir bemerken, wenn alle charakteristischen Exponenten λ_k der Gleichung

$$\frac{dp}{dt} = A(t)\, p \tag{1.15}$$

negative Realteile haben:

$$\max_{1\le k\le n} \operatorname{Re} \lambda_k < -\gamma < 0\,, \tag{1.16}$$

gilt die Abschätzung

$$\|P(t)\, P^{-1}(s)\| \le m e^{-\gamma(t-s)} \quad (m = \text{const},\ t \ge s)\,, \tag{1.17}$$

wobei $P(t)$ die Fundamentalmatrix von (1.15) mit $P(\tau) = I$ ist.

Lemma. Die Ungleichung (1.6) gilt für jede Lösung y_t von (1.14) zu jeder Anfangsbedingung $(y_0, \tau) \in L^2(\Omega, \mathcal{U}, P) \times R^1$ wenn die Bedingung (1.16) und Voraussetzung

$$\int_{\tau}^{t} \|C_s\|\, ds \le \varepsilon\,(t-\tau) \quad \text{(f. s.)} \quad (t \ge \tau,\ \varepsilon < \gamma\, m^{-1}) \tag{1.18}$$

erfüllt sind.

Beweis. y_t sei eine beliebige Lösung von (1.14) zu einer Anfangsbedingung $(y_0, \tau) \in L^2(\Omega, \mathcal{U}, P) \times R^1$, dann gilt

$$y_t = P(t)\, y_0 + \int_{\tau}^{t} P(t)\, P^{-1}(s)\, C_s\, y_s\, ds\,. \tag{1.19}$$

Aus (1.19) und (1.17) erhalten wir

$$e^{\gamma t}\|y_t\| \leq m e^{\gamma\tau}\|y_0\| + m\int_\tau^t e^{\gamma s}\|y_s\|\,\|C_s\|\,ds\,,$$

und nach dem Gronwall—Bellman-Lemma ([4], [5]) folgt wegen (1.18)

$$e^{\gamma t}\|y_t\| \leq m e^{\gamma\tau}\|y_0\| \exp\left\{m\int_\tau^t \|C_s\|\,ds\right\} \leq m e^{\gamma\tau}\|y_0\| \exp\{m\,\varepsilon(t-\tau)\} \quad \text{(f. s.)}.$$

Damit haben wir

$$E\|y_t\|^2 \leq m^2 E\|y_0\|^2 \exp\{2(m\,\varepsilon-\gamma)(t-\tau)\}\,,$$

d. h. in diesem Fall gilt (1.6) mit $\alpha = m^2$, $\varrho = 2(\gamma - m\,\varepsilon) > 0$ (wegen (1.18)). Das Lemma ist bewiesen.

Wegen $A_t = EA_t + (A_t - EA_t)$ können wir dieses Lemma auf (1.5) anwenden.

Aus Satz 1.2 und diesem Lemma folgt unmittelbar der folgende Satz für das System

$$\frac{d\,\xi_t}{dt} = [A(t) + C_t]\,\xi_t + z_t + \zeta_t\,, \tag{1.20}$$

wobei $A(t)$, C_t wie bei (1.14) sind, und z_t ein von C_t unabhängiger auf R^1 R-stetiger streng Θ-periodischer Prozeß zweiter Ordnung mit (1.2) und ζ_t ein von C_t unabhängiger auf R^1 R-stetiger Prozeß zweiter Ordnung mit $E\|\zeta_t\|^2 \in C(R^1)$ und $\sup_{t\in R^1} E\|\zeta_t\|^2 < \infty$ ist.

Satz 1.3. *Es seien die Voraussetzungen* (1.16), (1.18) *und* (1.8) *erfüllt. Dann ist jede Lösung* ξ_t *von* (1.20) *zu einer Anfangsbedingung* $(\xi_0, t_0) \in L^2(\Omega, \mathcal{U}, P) \times R^1$, *wobei* ξ_0 *vom Prozeß* (C_t, z_t) *unabhängig ist, asymptotisch* Θ*-periodisch.*

2. In [1] (S. 120), [2] hat H. Bunke eine Differentialgleichung der Gestalt

$$\frac{dx_t}{dt} = f(x_t, t, z_t) + g(x_t, t, z_t) \tag{2.1}$$

betrachtet. Der folgende Satz stellt eine Verallgemeinerung eines Resultates von A. Ja. Dorogovcev in [3] dar.

Satz 2.1 (H. Bunke [1], [2]). *Es seien folgende Voraussetzungen erfüllt:*

(i) $f(x, t, z)$ *und* $g(x, t, z)$ *sind auf* $R^n \times R^1 \times R^m$ *stetige n-dimensionale Vektorfunktionen, die für jedes feste* $(x, z) \in R^n \times R^m$ Θ*-periodisch sind. Es sei* $f(0, t, z) = 0$, $(t, z) \in R^1 \times R^m$.

(ii) z_t *ist ein R-stetiger streng* Θ*-periodischer m-dimensionaler Vektorprozeß.*

(iii) *Es gilt* $\|g(x, t, z_t)\| \leq \varphi(t, z_t)$ (f. s.) *mit*

$$\int_0^\infty E\varphi(\tau, z_\tau)\,d\tau = K < \infty \quad \textit{und} \quad \|g(x_1, t, z_t) - g(x_2, t, z_t)\| \leq \beta \|x_1 - x_2\| \quad (\text{f. s.}),$$

wobei β *hinreichend klein ist.*

(iv) *Es gibt eine reelle stetige* Θ*-periodische Matrixfunktion* $A_0(t)$*, deren charakteristische Exponenten negative Realteile haben:* $\max\limits_{1 \leq k \leq n} \operatorname{Re} \lambda_k < -\varrho < 0$*, so daß*

$$\|f(x_1, t, z_t) - f(x_2, t, z_t) - A_0(t)(x_1 - x_2)\| \leq \chi \|x_1 - x_2\| \quad (\text{f. s.})$$

mit hinreichend kleinem χ *gilt.*

Dann existiert auf R^1 *eine Lösung* x_t^0 *von* (2.1)*, für die* (x_t^0, z_t) *streng* Θ*-periodisch ist, und jede Lösung* x_t *von* (2.1) *konvergiert fast sicher exponentiell gegen diese streng periodische Lösung.*

Wegen Voraussetzung (iv) haben wir die Abschätzung

$$\|Q_0(t)\,Q_0^{-1}(\tau)\| \leq \gamma\, e^{-\varrho(t-\tau)} \quad (\gamma = \text{const},\ t \geq \tau), \tag{2.2}$$

wobei $Q_0(t)$ die Fundamentalmatrix von $\dot{q} = A_0(t)\,q$ mit $Q_0(0) = I$ ist. Im Beweis des Satzes 2.1 hat H. BUNKE gezeigt, daß

$$\lim_{t \to \infty} \left(e^{\varepsilon t} \|x_t - x_t^0\|\right) = 0 \quad (\text{f. s.}) \tag{2.3}$$

mit $0 < \varepsilon < \varrho - \gamma(\chi + \beta)$ gilt.

Nun betrachten wir das asymptotische Verhalten von Lösungen des Systems

$$\frac{d\xi_t}{dt} = f(\xi_t, t, z_t) + g(\xi_t, t, z_t) + h(\xi_t, t, z_t), \tag{2.4}$$

wobei $h(x, t, z)$ nichtperiodisch in t und asymptotisch klein ist.

SATZ 2.2. *Es seien die Bedingungen* (i), (ii), (iii), (iv) *des Satzes 2.1 und folgende Voraussetzung erfüllt:*

(v) $h(x, t, z)$ *ist eine auf* $R^n \times R^1 \times R^m$ *stetige* n*-dimensionale Vektorfunktion, so daß es* $\|h(x, t, z_t)\| \leq \psi(t, z_t)$ (f. s.) *mit* $\psi \in C(R^1 \times R^m)$ *und*

$$\lim_{t \to \infty} \left(e^{\delta t} \psi(t, z_t)\right) = 0 \quad (\text{f. s.}),$$

$\delta > 0$*, gilt.*

Dann konvergiert jede Lösung ξ_t *von* (2.4) *fast sicher exponentiell gegen die streng* Θ*-periodische Lösung* x_t^0 *von* (2.1)*, d. h.*

$$\lim_{t \to \infty} \left(e^{\varepsilon t} \|\xi_t - x_t^0\|\right) = 0 \quad (\text{f. s.}) \tag{2.5}$$

mit $0 < \varepsilon < \min\,[\delta, \varrho - \gamma(\chi + \beta)]$ *gilt.*

Wenn nur $\lim_{t\to\infty} \psi(t, z_t) = 0$ (f. s.) *erfüllt ist, dann gilt*

$$\lim_{t\to\infty} \|\xi_t - x_t^0\| = 0 \quad \text{(f. s.)}.$$

BEWEIS. Nach dem Satz 2.1 von H. BUNKE ist (2.5) bewiesen, wenn für jede Lösung ξ_t von (2.4) und jede Lösung x_t von (2.1)

$$\lim_{t\to\infty} (e^{\varepsilon t} \|\xi_t - x_t\|) = 0 \quad \text{(f. s.)} \tag{2.6}$$

mit $0 < \varepsilon < \min[\delta, \varrho - \gamma(\chi + \beta)]$ gilt.

Aus der Gleichung

$$\frac{d[\xi_t - x_t]}{dt} = A_0(t)[\xi_t - x_t] + f(\xi_t, t, z_t) - f(x_t, t, z_t) -$$
$$- A_0(t)[\xi_t - x_t] + g(\xi_t, t, z_t) - g(x_t, t, z_t) + h(\xi_t, t, z_t)$$

erhalten wir wegen (iii), (iv), (v) und (2.2)

$$\|\xi_t - x_t\| \le \|Q_0(t)\| \, \|\xi_0 - x_0\| + \chi \int_0^t \|Q_0(t) Q_0^{-1}(\tau)\| \, \|\xi_\tau - x_\tau\| \, d\tau +$$
$$+ \beta \int_0^t \|Q_0(t) Q_0^{-1}(\tau)\| \, \|\xi_\tau - x_\tau\| \, d\tau + \int_0^t \|Q_0(t) Q_0^{-1}(\tau)\| \, \|h(\xi_\tau, \tau, z_\tau)\| \, d\tau \le$$
$$\le \gamma \|\xi_0 - x_0\| e^{-\varrho t} + \gamma(\chi + \beta) \int_0^t e^{-\varrho(t-\tau)} \|\xi_\tau - x_\tau\| \, d\tau +$$
$$+ \gamma \int_0^t e^{-\varrho(t-\tau)} \psi(\tau, z_\tau) \, d\tau \quad \text{(f. s.)},$$

bzw.

$$e^{\varrho t} \|\xi_t - x_t\| \le \gamma \|\xi_0 - x_0\| + \gamma(\chi + \beta) \int_0^t e^{\varrho \tau} \|\xi_\tau - x_\tau\| \, d\tau +$$
$$+ \gamma \int_0^t e^{\varrho \tau} \psi(\tau, z_\tau) \, d\tau \quad \text{(f. s.)}.$$

Daraus erhalten wir nach dem verallgemeinerten Gronwall—Bellman-Lemma ([1], [6])

$$\|\xi_t - x_t\| \le \gamma \|\xi_0 - x_0\| e^{[\gamma(\chi+\beta) - \varrho]t} +$$
$$+ \gamma e^{[\gamma(\chi+\beta) - \varrho]t} \int_0^t e^{[\varrho - \gamma(\chi+\beta)]\tau} \psi(\tau, z_\tau) \, d\tau \quad \text{(f. s.)}. \tag{2.7}$$

Aus der Ungleichung

$$e^{\varepsilon t}\|\xi_t - x_t\| \leq \gamma\, \|\xi_0 - x_0\|\, e^{[\gamma(\chi+\beta)-\varrho+\varepsilon]t} +$$

$$+\gamma\, e^{[\gamma(\chi+\beta)-\varrho+\varepsilon]t} \int_0^t e^{[\varrho-\gamma(\chi+\beta)]\tau}\, \psi(\tau, z_\tau)\, d\tau \quad \text{(f. s.)}$$

erhalten wir (2.6) wegen (v) mit der L'Hospital-Regel.

Aus (2.7) folgt mit der L'Hospital-Regel

$$\lim_{t\to\infty} \|\xi_t - x_t\| = 0 \quad \text{(f. s.)}, \tag{2.8}$$

wenn $\lim_{t\to\infty} \psi(t, z_t) = 0$ (f. s.) gilt. Aus (2.3) und (2.8) erhalten wir $\lim_{t\to\infty} \|\xi_t - x_t^0\| = 0$ (f. s.). Der Satz 2.2 ist bewiesen.

Aus Satz 2.1 folgt unmittelbar das entsprechende Resultat für die Gleichung

$$\frac{dx_t}{dt} = f(x_t, z_t) + g(x_t, z_t). \tag{2.9}$$

Folgerung 2.1 (H. Bunke [1], [2]). Es seien folgende Voraussetzungen erfüllt:

(i) $f(x, z)$ und $g(x, z)$ sind auf $R^n \times R^m$ stetige n-dimensionale Vektorfunktionen. Es sei $f(0, z) = 0$, $z \in R^m$.

(ii) z_t ist ein R-stetiger streng stationärer m-dimensionaler Vektorprozeß.

(iii) Es gilt $\|g(x, z_t)\| < q(z_t)$ (f. s.) mit $E\,q(z_t) < \infty$ und $\|g(x_1, z_t) - g(x_2, z_t)\| \leq \beta \|x_1 - x_2\|$ (f. s.), wobei β hinreichend klein ist.

(iv) Es gibt eine reelle Matrix A_0, deren charakteristische Zahlen negative Realteile haben:

$$\max_{1\leq k\leq n} \operatorname{Re} \lambda_k < -\varrho < 0,$$

so daß

$$\|f(x_1, z_t) - f(x_2, z_t) - A_0(x_1 - x_2)\| < \chi\, \|x_1 - x_2\| \quad \text{(f. s.)}$$

mit hinreichend kleinem χ gilt.

Dann existiert auf R^1 eine Lösung x_t^0 von (2.9), für die (x_t^0, z_t) streng stationär ist, und jede Lösung x_t von (2.9) konvergiert fast sicher exponentiell gegen diese streng stationäre Lösung.

Aus Satz 2.2 folgt unmittelbar das folgende Resultat für die Gleichung

$$\frac{d\,\xi_t}{dt} = f(\xi_t, z_t) + g(\xi_t, z_t) + h(\xi_t, t, z_t). \tag{2.10}$$

Folgerung 2.2. Es seien die Voraussetzungen (i), (ii), (iii), (iv) der Folgerung 2.1 und die Voraussetzung (v) des Satzes 2.2. erfüllt. Dann konvergiert jede Lösung ξ_t von (2.10) fast sicher exponentiell gegen die streng stationäre Lösung x_t^0 von (2.9), d. h.

$$\lim_{t\to\infty} (e^{\varepsilon t} \|\xi_t - x_t^0\|) = 0 \quad \text{(f. s.)}$$

mit $0 < \varepsilon < \min[\delta, \varrho - \gamma(\chi + \beta)]$ gilt.

Wenn nur $\lim_{t\to\infty} \psi(t, z_t) = 0$ (f. s.) erfüllt ist, dann gilt

$$\lim_{t\to\infty} \|\xi_t - x_t^0\| = 0 \quad \text{(f. s)}.$$

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ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНО-РАЗНОСТНЫХ СИСТЕМ НЕЙТРАЛЬНОГО ТИПА С ИМПУЛЬСНЫМ ВОЗДЕЙСТВИЕМ

М. А. АРОЛСКА и Д. Д. БАЙНОВ

Пловдивский Университет имени П. Хилендарского

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В настоящей работе рассматривается задача о существовании периодических решений нелинейных систем дифференциально-разностных уравнений с импульсным воздействием вида

$$\dot{x}(t) = A\dot{x}(t-h) + Bx(t-h) + Cx(t) + f(t, x(t), x(t-h), \dot{x}(t-h)), \quad t \neq ih \tag{1}$$

$$\Delta x|_{t=ih} = I_i(x(t), x(t-h), \dot{x}(t-h))|_{t=ih-0},$$

где $x \in \mathbf{R}^m$, A, B, C-постоянные $(m \times m)$ матрицы, h-положительная постоянная, $\Delta x|_{t=ih} = x(ih+0) - x(ih-0)$, $(i = 0, \pm 1, \ldots)$, $f(t, x, y, z)$-периодическая по t с периодом T m-мерная вектор-функция, определенная на множестве $I \times D$, где

$$I = \{t : -\infty < t < +\infty\}, \tag{2}$$

$$D = \{(x, y, z) : \|x\| \leq \mu, \|y\| \leq \mu, \|z\| \leq \mu\},$$

μ-положительная постоянная, а период T имеет вид

$$T = ph, \tag{3}$$

где p-натуральное число; $I_i(x, y, z)$, $(i = 0, \pm 1, \ldots)$ — m — мерные вектор-функции, определенные на множестве D и удовлетворяющие равенствам

$$I_{i+p}(x, y, z) = I_i(x, y, z) \tag{4}$$

при $(x, y, z) \in D$.

Надо отметить, что аналогичные вопросы, но для систем обыкновенных дифференциальных уравнений рассмотрены в [1].

Будем говорить, что матрица A удовлетворяет условию (P) если ее собственные значения λ_i удовлетворяют условию $|\lambda_i| < 1$. Отметим, что тогда матричный ряд $\sum_{K=0}^{\infty} A^K$ сходится к $(E-A)^{-1}$ и справедливо неравенство

(5) $$\|(E-A)^{-1}\| \leq \sum_{K=0}^{\infty} \|A^K\| = L < \infty .$$

Будем говорить, что квадратная матрица H удовлетворяет условию (Q), если вещественные части ее собственных чисел отличны от нуля.

Пусть матрица H удовлетворяет условию (Q). Без ограничения общности можно предполагать, что $H = \operatorname{diag}(H_+, H_-)$, где H_+-матрица, собственные значения которой имеют положительные вещественные части, а H_--матрица, вещественные части собственных значений которой отрицательны.

Определим матрицу $G(t)$ соотношением

(6) $$G(t) = \begin{cases} \operatorname{diag}(-e^{-H_+t},\ 0) & \text{при} \quad t > 0 \\ \operatorname{diag}(\ \ 0,\ e^{-H_-t}) & \text{при} \quad t < 0 . \end{cases}$$

Как известно, существуют положительные постоянные K и γ такие, что

(7) $$\|G(t)\| \leq K e^{-\gamma|t|}$$

при $t \in I$.

Рассмотрим линейную систему

(8) $$\begin{aligned} &\dot{x}(t) = A\dot{x}(t-h) + Bx(t-h) + Cx(t) + g(t), \quad t \neq ih , \\ &\Delta x|_{t=ih} = I_i . \end{aligned}$$

Лемма 1. Пусть выполнены следующие условия:

1. Матрица A удовлетворяет условию (P).
2. Матрица $H = (E-A)^{-1}(B+C)$ удовлетворяет условию (Q).
3. Функция $g(t)$ кусочно непрерывна при $t \in I$ с точками разрыва первого рода при $t = ih$, ($i = 0, \pm 1, \ldots$) и периодическая с периодом $T = ph$.
4. Постоянные m-мерные векторы I_i ($i = 0, \pm 1, \ldots$) удовлетворяют равенствам $I_{i+p} = I_i$.
5. Выполнено неравенство

(9) $$\frac{4KTL_1\sqrt{m}}{\gamma}\left(1 + L_1 + \frac{\gamma}{2K}\right) < 1 ,$$

где $L_1 = L(\|B\| + \|C\|)$.

Тогда система (8) имеет единственное T-периодическое решение $x_T(t)$. Это решение удовлетворяет неравенству

(10) $$\|x_T(t)\|_1 \leq c \max\left\{\sup_{0 \leq t \leq T} \|g(t)\|,\ \sup_i \|I_i\|\right\} ,$$

где c-положительная постоянная, независящая от t, а через $\|x_T(t)\|_1$ обозначено

$$\|x_T(t)\|_1 = \sup_{\substack{0\le t\le T\\ t\ne ih}} \|x_T(t)\| + \sup_{\substack{0\le t\le T\\ t\ne ih}} \|\dot{x}_T(t)\| + \sup_i \|\Delta x_T|_{t=ih}\| + \sup_i \|\Delta \dot{x}_T|_{t=ih}\| .$$

Доказательство. Обозначим через M множество всех m-мерных T-периодических функций $w(t)$, определенных и непрерывно-дифференцируемых в I за исключением быть может точек $t = ih$ $(i = 0, \pm 1, \ldots)$, в которых $w(t)$ и $\dot{w}(t)$ могут иметь разрыв первого рода.

Множество M является польным нормированным пространством с нормой

$$\|w(t)\|_1 = \sup_{\substack{0\le t\le T\\ t\ne ih}} \|w(t)\| + \sup_{\substack{0\le t\le T\\ t\ne ih}} \|\dot{w}(t)\| + \sup_i \|\Delta w|_{t=ih}\| + \sup_i \|\Delta \dot{w}|_{t=ih}\| . \tag{11}$$

Для элементов $w(t)\in M$ определим линейные операторы A_h и B_h следующим образом

$$A_h w(t) = A\, w(t-h), \quad B_h w(t) = B\, w(t-h) . \tag{12}$$

Из условия 1 леммы 1 следует, что обратный оператор $(E-A_h)^{-1}$ существует и определяется формулой

$$(E-A_h)^{-1} = \sum_{K=0}^{\infty} A_h^K . \tag{13}$$

Кроме того имеет место оценка

$$\|(E-A_h)^{-1}\| \le L < \infty . \tag{14}$$

Пользуясь операторами A_h и B_h запишем систему (8) в виде

$$\begin{aligned} &\dot{x}(t) = (E-A)^{-1}(B+C)\,x(t) + (E-A_h)^{-1} g(t) + \tau(x, h), \quad t\ne ih .\\ &\Delta x|_{t=ih} = I_i , \end{aligned} \tag{15}$$

где положено

$$\tau(x, h) = [(E-A_h)^{-1} - (E-A)^{-1}]\,(B+C)\,x(t) + (E-A_h)^{-1}(B_h - B)\,x(t) . \tag{16}$$

Для исследования периодических решений систем типа (15) будем пользоваться следующей леммой:

Лемма 2. [*I*] Пусть система уравнений

$$\begin{aligned} &\dot{x}(t) = H\,x(t) + P(t), \quad t\ne t_i ,\\ &\Delta x|_{t=t_i} = I_i \end{aligned} \tag{17}$$

удовлетворяет условиям:

1. Матрица H удовлетворяет условию (Q).
2. Функция $P(t)$ кусочно непрерывна с точками разрыва первого рода при $t = t_i$ $(i = 0, \pm 1, \dots)$ и T-периодическая.
3. Для некоторого натурального числа p выполнено равенство

$$I_{i+p} = I_i .$$

4. Последовательность моментов t_i занумерована множеством целых чисел так, что $t_i \to -\infty$, $t_i \to +\infty$, $t_{i+p} = t_i + T$ и можно указать такое положительное число Θ, что $t_{i+1} - t_i \geq \Theta$ при $i = 0, \pm 1, \dots$.

Тогда система уравнений (17) имеет единственное периодическое решение $x^*(t)$, удовлетворяющее неравенству

$$\|x^*(t)\| \leq \frac{2K}{\gamma} \sup_{0 \leq t \leq T} \|P(t)\| + \frac{2K}{1 - e^{-\gamma\Theta}} \sup_i \|I_i\| , \tag{18}$$

где числа K и γ определены из (7).

Обозначим через $w_0(t)$ элемент множества M, для которого

$$\Delta w_0|_{t=ih} = I_i . \tag{19}$$

Рассмотрим систему

$$\begin{aligned} &\dot{x}_0(t) = (E - A)^{-1}(B + C)\, x_0(t) + (E - A_h)^{-1} g(t) + \tau(w_0, h), \quad t \neq ih , \\ &\Delta x_0|_{t=ih} = I_i . \end{aligned} \tag{20}$$

Система (20) удовлетворяет всем требованиям леммы 2 при

$$\begin{aligned} H &= (E - A)^{-1}(B + C) , \\ P(t) &= (E - A_h)^{-1} g(t) + \tau(w_0, h) , \\ t_i &= ih, \quad \Theta = h . \end{aligned} \tag{21}$$

Но тогда система (20) имеет единственное периодическое решение $x_0(t)$, удовлетворяющее неравенству (18), где H, $P(t)$ и Θ определены из (21).

Построим последовательность функций $x_n(t)$, каждая из которых является соответственно периодическим решением системы уравнений

$$\begin{aligned} &\dot{x}_n(t) = (E - A)^{-1}(B + C)\, x_n(t) + (E - A_h)^{-1} g(t) + \tau(x_{n-1}, h), \quad t \neq ih , \\ &\Delta x_n|_{t=ih} = I_i , \end{aligned} \tag{22}$$

взяв в качестве функции $x_0(t)$ решение системы (20).

Из леммы 2 следует, что при $n = 1, 2, \dots$, система (22) имеет единственное решение $x_n(t) \in M$.

Оценим норму $\|\tau(w, h)\|$ при $w \in M$. Имеем

$$\begin{aligned} \|\tau(w, h)\| \leq &\sum_{k=0}^{\infty} \|A^k (B + C)[w(t - kh) - w(t)]\| + \\ &+ \sum_{k=0}^{\infty} \|A^k B [w(t - (k+1)h) - w(t - kh)]\| . \end{aligned} \tag{23}$$

Обозначим через $w_j(t)$ j-тую компоненту вектора $w(t)$. Рассмотрим разность $w_j(t-kh)-w_j(t)$. Поскольку $w_j(t)$ T-периодическая функция, для каждого k можно указать такое натуральное число q, чтобы выполнялось равенство

$$w_j(t-kh) = w_j(t+qT-kh), \quad k \le p(q+1). \tag{24}$$

Из (24) следует

$$w_j(t-kh)-w_j(t) = w_j(t+qT-kh)-w_j(t),$$

где $t-(t+qT-kh) \le T$.

Для каждого k и соответствующего ему q на сегменте $[t+qT-kh, t]$ существуют не больше чем $(p+1)$ точки вида lh, где $l \in \{0, \pm 1, \ldots\}$. Применяем теорему Лагранжа для функции $w_j(t)$ на сегменте $[a, b]$, где через a обозначена некоторая из точек $t+qT-kh$, $(l+1)h, \ldots, (l+p)h$, а через b — соответственно некоторая из точек $(l+1)h, \ldots, (l+p)h, t$. Получаем

$$w_j(b)-w_j(a) = \dot w_j(t_j^*)(b-a), \quad t_j^* \in (a, b),$$

откуда следует оценка

$$\|w(b)-w(a)\| \le \sqrt{m} \sup_{\substack{0 \le t \le T \\ t \ne ih}} \|\dot w(t)\| h.$$

Дальше находим

$$\|w(t-kh)-w(t)\| \le p\sqrt{m}\left(\sup_{\substack{0 \le t \le T \\ t \ne ih}} \|\dot w(t)\| h + \sup_i \|\Delta w|_{t=ih}\|\right). \tag{25}$$

Аналогичным образом для выражения $\|w(t-(k+1)h)-w(t-kh)\|$ получаем

$$\|w(t-(k+1)h)-w(t-kh)\| \le \le 2\sqrt{m}\left(\sup_{\substack{0 \le t \le T \\ t \ne ih}} \|\dot w(t)\| h + \sup_i \|\Delta w|_{t=ih}\|\right). \tag{26}$$

Из (23), (25) и (26) следует оценка

$$\|\tau(w, h)\| \le 2pL_1\sqrt{m}\left(\sup_{\substack{0 \le t \le T \\ t \ne ih}} \|\dot w(t)\| h + \sup_i \|\Delta w|_{t=ih}\|\right). \tag{27}$$

Оценим теперь норму $\|x_{n+1}(t)-x_n(t)\|_1$ при $n = 1, 2, \ldots$.

Функция $x_{n+1}(t)-x_n(t)$ является решением следующей системы уравнений

$$\dot\zeta(t) = (E-A)^{-1}(B+C)\zeta(t) + \tau(x_n(t)-x_{n-1}(t), h). \tag{28}$$

Дальше будем пользоваться неравенством (см. [2], стр. 360)

$$\|x_{n+1}(t)-x_n(t)\| \le \frac{2K}{\gamma} \sup_{0 \le t \le T} \|\tau(x_n(t)-x_{n-1}(t), h)\|. \tag{29}$$

Из (27) и (29) находим оценку

$$\|x_{n+1}(t)-x_n(t)\|\leq\frac{4KTL_1\sqrt{m}}{\gamma}\sup_{0\leq t\leq T}\|\dot{x}_n(t)-\dot{x}_{n-1}(t)\|. \tag{30}$$

Из (28) получаем

$$\|\dot{x}_{n+1}(t)-\dot{x}_n(t)\|\leq L_1\|x_{n+1}(t)-x_n(t)\|+ \tag{31}$$
$$+2pL_1\sqrt{m}\,h\sup_{0\leq t\leq T}\|\dot{x}_n(t)-\dot{x}_{n-1}(t)\|.$$

Из (30) и (31) следует неравенство

$$\|x_{n+1}(t)-x_n(t)\|_1\leq R\,\|x_n(t)-x_{n-1}(t)\|_1, \tag{32}$$

где

$$R=\frac{4KTL_1\sqrt{m}}{\gamma}\left(1+L_1+\frac{\gamma}{2K}\right). \tag{33}$$

Из условия 5 леммы 1 следует сходимость последовательности $\{x_n(t)\}_{n=0}^{\infty}$ в пространстве M.

Введем обозначение $x_T(t)=\lim\limits_{n\to\infty}x_n(t)$. Функция $x_T(t)$ будет T-периодическим решением системы (8). Кроме того из системы (22) при $n\to\infty$ получаем

$$\begin{aligned}&\dot{x}_T(t)=(E-A)^{-1}(B+C)\,x_T(t)+(E-A_h)^{-1}g(t)+\tau(x_T(t),h),\quad t\neq ih,\\&\Delta x_T|_{t=ih}=I_i.\end{aligned} \tag{34}$$

Из леммы 2 следует оценка

$$\|x_T(t)\|\leq\frac{2K}{\gamma}\sup_{0\leq t\leq T}[(E-A_h)\,g(t)+\tau(x_T,h)]+\frac{2K}{1-e^{-\gamma h}}\sup_i\|I_i\|\leq$$
$$\leq\frac{2KL}{\gamma}\sup_{0\leq t\leq T}\|g(t)\|+\frac{2K}{\gamma}\sup_{0\leq t\leq T}\|\tau(x_T,h)\|+\frac{2K}{1-e^{-\gamma h}}\sup_i\|I_i\|.$$

Пользуясь оценкой (27) получаем

$$\|x_T(t)\|\leq\frac{2KL}{\gamma}\sup_{0\leq t\leq T}\|g(t)\|+\frac{4KTL_1\sqrt{m}}{\gamma}\sup_{\substack{0\leq t\leq T\\ t\neq ih}}\|\dot{x}_T(t)\|+ \tag{35}$$
$$+\left(\frac{4KTL_1\sqrt{m}}{\gamma h}+\frac{2K}{1-e^{-\gamma h}}\right)\sup_i\|I_i\|.$$

Из (34) при $t\neq ih$ находим

$$\|\dot{x}_T(t)\|\leq L_1\sup_{\substack{0\leq t\leq T\\ t\neq ih}}\|x_T(t)\|+L\sup_{\substack{0\leq t\leq T\\ t\neq ih}}\|g(t)\|+ \tag{36}$$
$$+\frac{2TL_1\sqrt{m}}{h}\left(\sup_{\substack{0\leq t\leq T\\ t\neq ih}}\|\dot{x}_T(t)\|\,h+\sup_i\|I_i\|\right).$$

Из (35) и (36) следует оценка

(37)
$$\sup_{\substack{0\le t\le T\\ t\ne ih}}\|\dot x_T(t)\|\le\left(\frac{2KLL_1}{\gamma}+L\right)\sup_{\substack{0\le t\le T\\ t\ne ih}}\|g(t)\|+$$
$$+\left(\frac{4KTL_1^2\sqrt{m}}{\gamma}+2TL_1\sqrt{m}\right)\sup_{\substack{0\le t\le T\\ t\ne ih}}\|\dot x_T(t)\|+$$
$$+\left[L_1\left(\frac{4KTL_1\sqrt{m}}{\gamma}+\frac{2K}{1-e^{-\gamma h}}\right)+\frac{2TL_1\sqrt{m}}{h}\right]\sup_i\|I_i\|.$$

Далее из (15) находим

(38)
$$\|\Delta x_T|_{t=ih}\|\le 5L_1\sup_i\|I_i\|+2L\sup_{0\le t\le T}\|g(t)\|.$$

Из (35), (37) и (38) получаем

$$\|x_T(t)\|_1\le 2TL_1\sqrt{m}\left(\frac{2K}{\gamma}+\frac{2KL_1}{\gamma}+1\right)\|x_T(t)\|_1+$$
$$+\left(\frac{2KL}{\gamma}+\frac{2KLL_1}{\gamma}+3L\right)\sup_{0\le t\le T}\|g(t)\|+$$
$$+\left(\frac{4KTL_1\sqrt{m}}{\gamma h}+\frac{2K}{1-e^{-\gamma h}}+\right.$$
$$\left.+L_1\left(5+\frac{2T\sqrt{m}}{h}+\frac{2K}{1-e^{-\gamma h}}+\frac{4KTL_1\sqrt{m}}{\gamma h}\right)\right)\sup_i\|I_i\|.$$

Из (33) и условия 5 леммы 1 находим

(39)
$$\|x_T(t)\|_1\le c\max\left\{\sup_{0\le t\le T}\|g(t)\|;\ \sup_i\|I_i\|\right\},$$

где

(40)
$$c=\frac{1}{1-R}\max\left\{L\left(\frac{2K}{\gamma}+\frac{2KL_1}{\gamma}+3\right);\right.$$
$$\left.\frac{4KTL_1\sqrt{m}}{\gamma h}+\frac{2K}{1-e^{-\gamma h}}+L_1\left(5+\frac{2T\sqrt{m}}{h}+\frac{2K}{1-e^{-\gamma h}}+\frac{4KTL_1\sqrt{m}}{\gamma h}\right)\right\}.$$

Следовательно периодическое решение $x_T(t)$ системы (8) удовлетворяет неравенству (10). Единственность этого решения следует из тех соображений, что поскольку величина толчков при каждом ih ($i = 0, \pm 1, \ldots$) для всех решений постоянна, то разность двух периодических решений $x_T(t)$ и $x_T^*(t)$ системы (8) $\Delta(t) = x_T(t) - x_T^*(t)$ будет периодическим решением системы

(41)
$$\dot\Delta(t) = A\dot\Delta(t-h)+B\Delta(t-h)+C\Delta(t).$$

Рассуждая как и при доказательстве неравенства (39) заключаем, что $\Delta(t)$ будет удовлетворять неравенству типа (39) при $g(t)\equiv 0$ и $I_i\equiv 0$, $(i=0, \pm 1, \dots)$. Следовательно, $\Delta(t)=0$.

Этим лемма 1 доказана.

Используя лемму 1 можно доказать существование единственного периодического решения системы (1). Точнее, имеет место следующая теорема:

Теорема 1. *Пусть выполнены следующие условия:*

1. *Матрица A удовлетворяет условию* (P).
2. *Матрица* $(E-A)^{-1}(B+C)$ *удовлетворяет условию* (Q).
3. *Функции* $f(t, x, y, z)$ *и* $I_i(x, y, z)$, $(i=0, \pm 1, \dots)$ *определены и непрерывны по своим аргументам в множествах* $I\times D$ и D *соответственно, удовлетворяют условию Липшица по* x, y, z *равномерно по* t *и* i *с постоянной* N, *т. е.,*

$$\|f(t, x_1, y_1, z_1)-f(t, x_2, y_2, z_2)\| \le \le N(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|), \tag{42}$$

$$\|I_i(x_1, y_1, z_1)-I_i(x_2, y_2, z_2)\| \le N(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|)$$

при $t\in I$, (x_1, y_1, z_1), $(x_2, y_2, z_2)\in D$ *и равенствам*

$$\begin{aligned} &f(t+T, x, y, z)=f(t, x, y, z) \quad \text{при} \quad T=ph, \ (t, x, y, z)\in I\times D \\ &I_{i+p}(x, y, z)=I_i(x, y, z) \quad \text{при} \quad (x, y, z)\in D. \end{aligned} \tag{43}$$

4. *Выполнено неравенство* (9).

Тогда при достаточно малых значениях постоянной Липшица N *система 1 имеет единственное* T*-периодическое решение.*

Доказательство. Пусть $x_0(t)$ T-периодическое решение системы

$$\begin{aligned} &\dot{x}_0(t)=A\dot{x}_0(t-h)+Bx_0(t-h)+Cx_0(t)+f(t, 0, 0, 0), \quad t\ne ih, \\ &\Delta x_0|_{t=ih}=I_i(0, 0, 0), \end{aligned} \tag{44}$$

а функции $x_n(t)$ $(n=1, 2, \dots)$ определены из систем

$$\begin{aligned} &\dot{x}_n(t)=A\dot{x}_n(t-h)+Bx_n(t-h)+Cx_n(t)+ \\ &+f\big(t, x_{n-1}(t), x_{n-1}(t-h), \dot{x}_{n-1}(t-h)\big), \quad t\ne ih, \\ &\Delta x_n|_{t=ih}=I_i\Big(x_{n-1}(ih-0), x_{n-1}\big((i-1)h-0\big), \dot{x}_{n-1}\big((i-1)h-0\big)\Big). \end{aligned} \tag{45}$$

Согласно леммы 1, системы (44) и (45) имеют единственные решения.

Введем обозначение

$$S=\max\Big\{\sup_{0\le t\le T}\|f(t, 0, 0, 0)\|, \ \sup_i\|I_i(0, 0, 0)\|\Big\}. \tag{46}$$

Докажем, что при достаточно малых значениях N справедливы неравенства

$$\|x_n(t)\| \leq \frac{c\,S}{1-3\,N\,c}, \tag{47}$$

$$\|x_{n-1}(t)-x_n(t)\| \leq \frac{(3\,N\,c)^n}{1-3\,N\,c}\,c\,S, \quad (n=0,\,1,\,\ldots), \tag{48}$$

где постоянная c определена из (40).

Действительно, для функции $x_0(t)$ из леммы 1 следует оценка

$$\|x_0(t)\|_1 \leq c\,S, \tag{49}$$

откуда следует справедливость неравенства (47) при $n=0$, если постоянная Липшица N выбрана настолько малой, чтобы

$$3\,N\,c < 1. \tag{50}$$

При $n=1$ применяя лемму 1 получаем

$$\|x_1(t)\|_1 \leq c \max\Big\{\sup_{0\leq t\leq T}\|f(t,\,x_0(t-0),\,x_0(t-h-0),\,\dot{x}_0(t-h-0))\|,$$
$$\sup_i\Big\|I_i\big(x_0(ih-0),\,x_0((i-1)\,h-0),\,\dot{x}_0((i-1)\,h-0)\big)\Big\|\Big\} \leq$$
$$\leq c\max\Big\{\sup_{0\leq t\leq T}\|f(t,\,x_0(t-0),\,x_0(t-h-0),\,\dot{x}_0(t-h-0))-f(t,\,0,\,0,\,0)\|+$$
$$+\sup_{0\leq t\leq T}\|f(t,\,0,\,0,\,0)\|,$$
$$\sup_i\Big\|I_i\big(x_0(ih-0),\,x_0((i-1)\,h-0),\,\dot{x}_0((i-1)\,h-0)\big)-I_i(0,\,0,\,0)\Big\|+$$
$$+\sup_i\|I_i(0,\,0,\,0)\|\Big\} \leq c\max\Big\{2N\sup_{0\leq t\leq T}\|x_0(t-0)\|+N\sup_{0\leq t\leq T}\|\dot{x}_0(t-0)\|+$$
$$+\sup_{0\leq t\leq T}\|f(t,\,0,\,0,\,0)\|,\;2N\sup_{0\leq t\leq T}\|x_0(t-0)\|+$$
$$+N\sup_{0\leq t\leq T}\|\dot{x}_0(t-0)\|+\sup_i\|I_i(0,\,0,\,0)\|\Big\}.$$

Имея ввиду (49) окончательно получаем

$$\|x_1(t)\|_1 \leq c\,S\,(1+3\,N\,c). \tag{51}$$

Аналогичным образом устанавливается справедливость неравенства

$$\|x_n(t)\|_1 \leq c\,S\,\big(1+3\,N\,c+\ldots+(3\,N\,c)^n\big), \quad (n=1,\,2,\,\ldots). \tag{52}$$

Потребуем, чтобы постоянная μ удовлетворяла неравенству

$$\frac{c\,S}{1-3\,N\,c} \leq \mu. \tag{53}$$

Тогда каждая из функций $x_n(t)$, $(n=1,\,2,\,\ldots)$, принимает значения из области $\|x\| \leq \mu$ и удовлетворяет неравенству (47).

Докажем теперь неравенство (48). Положим

$$y_{n+1}(t) = x_{n+1}(t) - x_n(t), \quad (n = 0, 1, \ldots).$$

Пусть $n = 0$. Функция $y_1(t)$ будет удовлетворять системе

$$\dot{y}_1(t) = A\,\dot{y}_1(t-h) + B\,y_1(t-h) + C\,y_1(t) + f\big(t, x_0(t), x_0(t-h), \dot{x}_0(t-h)\big) - f(t, 0, 0, 0), \quad t \neq ih,$$

(54)

$$\Delta y_1|_{t=ih} = I_i\Big(x_0(ih-0), x_0\big((i-1)h-0\big), \dot{x}_0\big((i-1)h-0\big)\Big) - I_i(0, 0, 0).$$

Из леммы 1 и условия (42) следует

$$\|y_1(t)\|_1 \leq c\Big(2N \sup_{0 \leq t \leq T} \|x_0(t-0)\| + N \sup_{0 \leq t \leq T} \|\dot{x}_0(t-0)\|\Big).$$

Из (49) и (50) получаем

(55) $$\|y_1(t)\|_1 < 3Nc\,\frac{cS}{1-3Nc}.$$

Из (55) следует справедливость неравенства (48) при $n = 0$.

Методом индукции, аналогично как при доказательстве неравенства (55), доказывается неравенство (48) при $n = 1, 2, \ldots$.

Из неравенства (48) следует равномерная при $t \in I$ сходимость последовательности $\{x_n(t)\}_{n=0}^{\infty}$. Предельная функция будет T-периодическим решением системы (1).

Докажем теперь единственность периодического решения системы (1). Допустим противное. Пусть $\varphi_1(t)$ и $\varphi_2(t)$ — два различных T-периодических решения этой системы. Функция $\psi(t) = \varphi_1(t) - \varphi_2(t)$ будет удовлетворять системе

$$\dot{\psi}(t) = A\,\dot{\psi}(t-h) + B\,\psi(t-h) + C\,\psi(t) + f\big(t, \varphi_1(t), \varphi_1(t-h), \dot{\varphi}_1(t-h)\big) - f\big(t, \varphi_2(t), \varphi_2(t-h), \dot{\varphi}_2(t-h)\big), \quad t \neq ih,$$

(56)

$$\Delta\psi|_{t=ih} = I_i\big(\varphi_1(t), \varphi_1(t-h), \dot{\varphi}_1(t-h)\big) - I_i\big(\varphi_2(t), \varphi_2(t-h), \dot{\varphi}_2(t-h)\big)\big|_{t=ih-0}.$$

Из леммы 1 и условий (42) следует оценка

$$\|\psi(t)\|_1 \leq 2cN\,\|\psi(t)\|_1.$$

Из неравенства (50) следует, что $\psi(t) \equiv 0$.

Этим теорема 1 доказана.

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ОЦЕНКА СНИЗУ СУММЫ КВАДРАТОВ ФУНДАМЕНТАЛЬНЫХ ФУНКЦИЙ ОДНОМЕРНОГО ОПЕРАТОРА ШРЕДИНГЕРА И ЕЁ ПРИМЕНЕНИЯ

И. ЙО

II. Кафедра Анализа Университета Л. Этвеша, Будапешт

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В настоящей работе изучается произвольная фундаментальная система функций (ФСФ) оператора

$$l = -\frac{d^2}{dx^2} + q(x), \tag{1}$$

на конечном или бесконечном интервале $G = (a, c)$. Предположим $q \in \in L^1_{\text{loc}}(G)$. Напомним определение ФСФ, впервые введенной В. А. Ильиным (см [1], [2]) и распространенной на случай оператора l в [3]. Полная ортонормированная в $L^2(G)$ система $\{u_n\}$ называется ФСФ оператора (1) на интервале G, если каждая функция u_n вместе со своей производной абсолютно непрерывна на каждом отрезке интервала G, $lu_n \in L^2(\Gamma)$ и для некоторого неотрицательного числа λ_n удовлетворяет уравнению

$$-\frac{d^2}{dx^2} u_n(x) + q(x) u_n(x) = \lambda_n u_n(x) \tag{2}$$

почти всюду в G. Заметим, что ФСФ обобщает понятие системы собственных функций отвечающих произвольным неотрицательным самосопряженным операторам с дискретным спектром, порожденных формальным дифференциальным оператором (1) (см. [3]).

Главный результат настоящей работы заключается в следующем утверждении:

Теорема.

а) *Для произвольной ФСФ оператора* (1) *на конечном интервале G с потенциалом $q \in L^p(G)$ $(p > 1)$ и для любого подотрезка I интервала G найдутся такие положительные числа M и α, что справедлива оценка:*

$$\sum_{|\sqrt{\lambda_n} - \mu| \leq M} |u_n(x)|^2 \geq \alpha > 0, \quad (x \in I, \ \mu > 0). \tag{3}$$

b) *Для произвольной* ФСФ *оператора* (1) *на конечном или бесконечном интервале* G *с потенциалом* $q \in L^2_{\text{loc}}(G)$ *и для любого подотрезка* I *интервала* G *найдутся такие положительные числа* M *и* α, *что справедлива* (3).

Из работы [3] и [7] следует, что при выполнении условий утверждения a) или b) справедлива следующая оценка сверху:

(4) $$\sum_{|\sqrt{\lambda_n}-\mu|\le 1} |u_n(x)|^2 = O(1), \quad (\mu \ge 0)$$

равномерно по x на любом компакте интервала G.

Суммы стоящие на левой части (3) или (4) соответственно могут содержать бесконечное число слагаемых (если числа $\{\lambda_n\}$ имеют конечные точки сгущения), но в этом случае эти ряды сходятся абсолютно. В работе [3] доказана, что при выполнении условий утверждения a) справедлива следующая оценка:

(5) $$\sum_{|\sqrt{\lambda_n}-\mu|\le 1} |u_n(x)|^2 \le C_1 \quad (x \in G, \ \mu \ge 0),$$

где постаянная C_1 не зависит от x и от μ. Интегрируя обе части (5) по x на интервале G получим:

(6) $$\sum_{|\sqrt{\lambda_n}-\mu|\le 1} 1 \le C_1 \cdot |G|, \quad (\mu \ge 0).$$

Отсюда вытекает, что в этом случае числа $\{\lambda_n\}$ не имеют конечные точки сгущения и таким образом можно предполагать, что они занумерованы в порядке неубывания. (Из (3) получим

(7) $$\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} = O(1), \quad (n = 1, 2, \ldots).)$$

С другой стороны из (6) сразу вытекает следующая оценка сверху для числа $\mathcal{N}(\lambda)$ собственных значений не превосходящих:

(8) $$\mathcal{N}(\lambda) \overset{\text{опр}}{=} \sum_{\lambda_n < \lambda} 1 \le C_2 \sqrt{\lambda}, \quad (\lambda \ge 0)$$

(см. [3], (6)).

В работе [3] доказана, что если G конечный или бесконечный интервал и $q \in L^p(G)$, $p > 1$, то имеет место оценка:

(9) $$\sum_{|\sqrt{\lambda_n}-\mu|\le 1} |u_n(x)|^q \le C, \quad (x \in G, \ \mu \ge 0).$$

Заметим наконец, что из оценки (3) вытекают:

(10) $$\sum_{\mu - M \le \sqrt{\lambda_n} \le \mu + M} 1 \ge \alpha \cdot |I|, \quad (\mu \ge 0)$$

и

(11) $$\mathcal{N}(\lambda) \ge C_3 \sqrt{\lambda} > 0, \quad (\lambda > 0)$$

и таким образом оценки (6) и (8) точные по порядку λ.

Следствие 1. *Из оценок* (3) *и* (6) *вытекает, что для каждой точки* $x \in G$ *существуют последовательность* $n_k = n_k(x)$ *натуральных чисел и такая положительная постоянная* $\beta = \beta(x)$ *для которых*

$$|u_{n_k(x)}(x)| \geq \beta(x) > 0, \quad (k = 1, 2, \ldots). \tag{12}$$

Постоянную $\beta(x)$ *можно выбрать одним и тем же на каждом компакте точек* x *интервала* G.

Таким образом оценка (12) справедлива при выполнении условий утверждения a) или b) (в случае выполнении условия утверждении b) надо дополнительно предполагать также выполнение (6)). Условия последнего утверждения выполняются например для классических ортогональных функций (функций Якоби, Лагерра и Эрмита) и ряда других специальных функций.

Следствие 2. *Используя оценку* (12) *можно доказать, что для каждой точки* $x \in G$ *существуют последовательность* $n_k = n_k(x)$ *натуральных чисел и такое число* $0 < y < \varrho(x, \partial G)$ *для которых*

$$\left| \int_{x-y}^{x+y} u_{n_k(x)}(t)\, dt \right| \geq C(x) \cdot \frac{1}{\sqrt{\lambda_{n_k(x)}}}, \quad (k = 1, 2, \ldots) \tag{13}$$

с некоторой положительной постоянной $C = C(x)$. *Постоянную* $C(x)$ *можно выбрать одним и тем же на каждом компакте точек* x *интервала* G.

В работе [6] доказано что при выполнении условий утверждения a) или b) справедлива оценка сверху:

$$\left| \int_{\sigma_1}^{\sigma_2} u_n(t)\, dt \right| \leq C(I) \frac{1}{\sqrt{\lambda_n}}, \quad (\lambda_n \geq 1;\ \sigma_1, \sigma_2 \in I) \tag{14}$$

для любого отрезка I интервала G. Постоянная $C = C(I)$ зависит только от отрезка I.

Следствие 3. *Пусть выполнены условия утверждения* b) *теоремы. Тогда для каждой точки* $x \in G$ *существуют последовательность* $\mu_k = \mu_k(x)$ *натуральных чисел и такое число* $0 < y < \varrho(x, \partial G)$ *для которых*

$$\sum_{\mu_k - M \leq \sqrt{\lambda_n} \leq \mu_k + M} \left| \int_{x-y}^{x+y} u_n(t)\, dt \right|^2 \geq C(x) \frac{1}{\mu_k}, \quad (k = 1, 2, \ldots) \tag{15}$$

с некоторой положительной постоянной $C = C(x)$. *Постоянную* $C(x)$ *можно выбрать одним и тем же на каждом компакте точек* x *интервала* G.

В работе [6] было доказано, что имеет место оценка

$$(16)\qquad \sum_{t\le\sqrt{\lambda_n}\le t+1}\Big|\int_{x_1}^{x_2} u_n(t)\,dt\Big|^2 = O\left(\frac{1}{(t+1)^2}\right),$$

в которой t — любое неотрицательное вещественное число, а x_1 и x_2 любые два числа из замкнутого отрезка $\bar G=[a,b]$ в случае а) и из любого фиксированного компакта интервала G — в случае b) теоремы. (x_1 и x_2 любые два числа из конечного или бесконечного интервала $\bar G$, в случае $q\in L^2(G)$).

Напомним, что результат аналогичный (3) для ФСФ оператора Лапласа даже в многомерном случае был получен В. А. Ильиным в [4]. Доказательство теоремы настоящей работы и её следствий основан на методе В. А. Ильина (см. [4]) а также некоторые результатах работ [3], [6] и [7].

Автор глубоко благодарит профессора В. А. Ильина за внимание к данной работе.

Доказательство теоремы и её следствий. Сперва докажем теорему. Ради удобства, разобъем доказательство теоремы на отдельные пункты. Сначала докажем утверждение а).

1. Фиксируем произвольный отрезок I интервала G, и некоторое положительное число $R>0$, меньше минимума расстояния между границами отрезка I и интервала G. Пусть x произвольная фиксированная точка отрезка I а μ любое положительное число. Рассмотрим функцию:

$$(17)\qquad w_R(|x-y|,\mu)=\begin{cases}\frac{1}{\pi}\cos|x-y| & \text{при}\quad |x-y|\le R\\ 0 & \text{при}\quad |x-y|>R\end{cases}.$$

Коэффициент Фурье w_n функции (13) по ФСФ $\{u_n\}$ подсчитан например в работе [7]. Он имеет вид

$$(18)\qquad w_n=u_n(x)\cdot\frac{2}{\pi}\int_0^R\cos\mu t\cos\sqrt{\lambda_n}\,t\,dt-$$

$$-\frac{1}{\pi\sqrt{\lambda_n}}\int_0^R\cos\mu t\int_{x-t}^{x+t}q(\xi)\,u_n(\xi)\sin\sqrt{\lambda_n}\,(|x-\xi|-t)\cdot d\xi\,dt.$$

(Используя тождественное преобразование «сдвиг потенциала»:

$$-u_n''(x)+[q(x)+2]\,u_n(x)=(\lambda_n+2)\,u_n(x)$$

и включение

$$\{n:|\sqrt{\lambda_n}-\mu|\le M\}\subset\{n:|\lambda_n+2-\mu|\le M+2\},$$

мы можем считать в данной работе, что $\lambda_n > 1$). Обозначим через M пока произвольное положительное число (выбор его произведем ниже) и запишем для функции (17) равенство Парсеваля, разбивая все коэффициенты Фурье на две группы, в первую из которых входят коэффициенты для которых $|\sqrt{\lambda_n} - \mu| \leq M$ а во вторую группу — со всеми остальными номерами. Будем иметь

$$(19) \qquad \int_a^b |w_R(|x-y|, \mu)|^2 \, dy = \sum_{|\sqrt{\lambda_n} - \mu| \leq M} |w_n|^2 + \sum_{|\sqrt{\lambda_n} - \mu| > M} |w_n|^2 .$$

Прежде всего убедимся в том, что при любом фиксированном R, для интеграла, стоящего в левой части (19), равномерно относительно x на отрезке I справедлива при больших μ следующая оценка:

$$(20) \qquad \int_a^b |w_R(|x-y|, \mu)|^2 \, dy \geq \frac{R}{2\pi^2} .$$

В самом деле, пользуясь тождеством $\cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \cos 2\alpha$, имеем

$$\int_a^b |w_R(|x-y|, \mu)|^2 \, dy = \frac{R}{\pi^2} + O\left(\frac{1}{\mu}\right).$$

2. Убедимся теперь в том, что для любого фиксированного числа M, при $\mu \to \infty$, равномерно относительно x на отрезке I, справедлива следующая оценка:

$$(21) \qquad \sum_{|\sqrt{\lambda_n} - \mu| \leq M} |w_n|^2 \leq R^2 \cdot \sum_{|\sqrt{\lambda_n} - \mu| \leq M} |u_n(x)|^2 + o(1) .$$

Из (18) в силу неравенства $(a+b)^2 \leq 2(a^2+b^2)$ вытекает:

$$\sum_{|\sqrt{\lambda_n} - \mu| \leq M} |w_n|^2 \leq R^2 \cdot \sum_{|\sqrt{\lambda_n} - \mu| \leq M} |u_n(x)|^2 +$$
$$+ 2 \sum_{|\sqrt{\lambda_n} - \mu| \leq M} \left| \frac{1}{\pi \sqrt{\lambda_n}} \int_0^R \cos \mu t \int_{x-t}^{x+t} q(\xi) \, u_n(\xi) \sin \sqrt{\lambda_n} \, (|x-\xi| - t) \, d\xi \, dt \right|^2 .$$

Нам осталось доказать оценку

$$(22) \qquad \sum_{|\sqrt{\lambda_n} - \mu| \leq M} |\hat{w}_n|^2 = o(1), \quad (\mu \to +\infty)$$

где

$$(23) \quad \hat{w}_n = \frac{1}{\pi \sqrt{\lambda_n}} \int_0^R \cos \mu t \int_{x-t}^{x+t} q(\xi) \, u_n(\xi) \sin \sqrt{\lambda_n} \, (|x-\xi| - t) \, d\xi \, dt .$$

Меняя порядок интегрирования в правой части (23) имеем:

$$\hat{w}_n = \frac{1}{\pi} \int_{x-R}^{x+R} q(\xi) \frac{u_n(\xi)}{\sqrt{\lambda_n}} \left[\int_{|x-\xi|}^{R} \cos \mu t \sin \sqrt{\lambda_n} \, (|x-\xi|-t)\, dt \right] d\xi . \tag{24}$$

Используя обозначение

$$\sigma_n = \int_{|x-\xi|}^{R} \cos \mu t \sin \sqrt{\lambda_n} \, (|x-\xi|-t)\, dt \tag{25}$$

оценку $\sigma_n = O(1)$ и неравенство Коши—Буняковского, получим:

$$|\hat{w}_n|^2 = O(1) \left(\int_{x-R}^{x+R} |q(\xi)|\, d\xi \right) \left(\int_{x-R}^{x+R} |q(\xi)| \frac{|u_n(\xi)|^2}{\lambda_n} d\xi \right) = \tag{26}$$

$$= O(1) \cdot \|q\|_{L^1(I_R)} \cdot \int_{I_R} |q(\xi)| \frac{|u_n(\xi)|^2}{\lambda_n} d\xi ,$$

где $I_R = \{x \in G : \varrho(x, I) \leq R\}$. Из (4) и (26) вытекает (22).

3. Из сопоставления равенства Парсеваля (19) с доказанными нами неравенствами (20) и (21) вытекает, что для доказательства требуемой оценки снизу (3) достаточно доказать, что можно фиксировать число M таким, что при всех достаточно больших μ, равномерно относительно x на отрезке I будет справедливо неравенство:

$$\sum_{|\sqrt{\lambda_n} - \mu| \geq M} |w_n|^2 < \frac{R}{4\pi^2} . \tag{27}$$

При доказательстве неравенства (27) мы будем опираться на оценку (4) и его легко доказываемые следствия:

Для любого $\mu \geq 1$ и любого ϱ из отрезка $1 \leq \varrho \leq \mu$ равномерно относительно x на каждом компакте интервала G справедлива оценка

$$\sum_{|\sqrt{\lambda_n} - \mu| \leq \varrho} |u_n(x)|^2 = \varrho \cdot O(1) . \tag{28}$$

Для любого $\delta > 0$, на каждом компакте интервала G, ряд

$$\sum_{n=1}^{\infty} \frac{|u_n(x)|^2}{\lambda_n^{(1/2)+\delta}} \tag{29}$$

обладает равномерно ограниченным семейством частичных сумм.

Для доказательства неравенства (27) разобьем сумму стоящую в левой части (27) на два суммы

(30) $$S_1 = \sum_{1 \le \sqrt{\lambda_n} < (\mu/2)} |w_n|^2 + \sum_{\sqrt{\lambda_n} > (3/2)\mu} |w_n|^2,$$

(31) $$S_2 = \sum_{M < |\sqrt{\lambda_n} - \mu| \le (\mu/2)} |w_n|^2.$$

Достаточно доказать, что можно фиксировать число M таким, что при всех достаточно больших μ каждая из сумм (30) и (31) для всех x из отрезка I не превзойдет величины $R/8\pi^2$.

4. Приступим теперь к оценке (30). В силу тождества

(32) $$\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\},$$

получаем

(33) $$\left|\frac{2}{\pi}\int_0^R \cos\mu t \cos\sqrt{\lambda_n}\, t\, dt\right| \le \frac{1}{|\sqrt{\lambda_n} - \mu|}, \quad (\sqrt{\lambda_n} \ne \mu).$$

Далее легко убедиться в том, что для всех номеров n, участвующих в сумме (30), справедливы неравенства $|\sqrt{\lambda_n} - \mu| > \frac{\mu}{2}$, $|\sqrt{\lambda_n} - \mu| > \frac{1}{3}\sqrt{\lambda_n}$. Из этих неравенств вытекает, что для всех номеров n, участвующих в сумме (30) справедлива оценка:

(34) $$\frac{1}{|\sqrt{\lambda_n} - \mu|} \le \frac{6}{\lambda_n^{3/8} \cdot \mu^{1/4}}.$$

В силу (33) и (34) имеем:

(35) $$S_1 \le 12\left\{\frac{1}{\sqrt{\mu}} \sum_{n=1}^{\infty} \frac{|u_n(x)|^2}{\lambda_n^{3/4}} + \sum_{|\sqrt{\lambda_n} - \mu| \ge (\mu/2)} |\hat{w}_n|^2\right\}.$$

Легко убедиться в том, что

(36) $$\sigma_n \le \frac{4}{|\sqrt{\lambda_n} - \mu|}, \quad (\sqrt{\lambda_n} \ne \mu).$$

Используя (36) и неравенство Коши—Буняковского, получим:

(37) $$\sum_{|\sqrt{\lambda_n} - \mu| \ge (\mu/2)} |\hat{w}_n|^2 \le \frac{8}{\mu^2} \|q\|_{L_1(I_R)I_R} \cdot \int_{I_R} |q(\xi)| \sum_{n=1}^{\infty} \frac{|u_n(\xi)|^2}{\lambda_n} d\xi.$$

Учитывая, что частичные суммы ряда (29) при $\delta > 0$ равномерно ограничены на каждом отрезке интервала G, получим из (35) и (37):

(38) $$S_1 \le \frac{R}{8\pi^2}, \quad (x \in I, \ \mu \ge \mu_0).$$

5. Для завершения доказательства утверждения a) теоремы, остается оценить сумму (31). Достаточно доказать, что фигурирующие в этой сумме число M можно фиксировать таким, что при всех $\mu>0$ сумма (31) для всех x из отрезка не превзойдет величины $R/8\pi^2$. Рассмотрим последовательность отрезков $[1,2]$, $[2,4]$, ..., $[2^{k-1},2^k]$ ... и обозначим через p наименьший из номеров k, для которых $2^k \geq \mu/2$. Без ограничения общности можно считать, что число M представляется в виде 2^{m-1}, где m — некоторый номер. Покажем, что номер m можно фиксировать таким, что для всех $\mu>0$, сумма (31) со значением $M=2^{m-1}$ для всех x из отрезка I не превзойдет величины $R/8\pi^2$.

В соответствии с принятыми обозначениями мы можем записать для суммы (31) следующее неравенство:

$$S_2 \leq 2\left\{\sum_{M<|\sqrt{\lambda_n}-\mu|\leq(\mu/2)} \frac{|u_n(x)|^2}{|\sqrt{\lambda_n}-\mu|^2} + \sum_{M<|\sqrt{\lambda_n}-\mu|\leq(\mu/2)} |\hat{v}_n|^2\right\}. \tag{39}$$

Здесь мы использовали неравенства (33) и $(a+b)^2 \leq 2(a^2+b^2)$. Из (39) в силу (33) и (36) получим:

$$S_2 \leq 2\left\{\sum_{k=m}^{p-1}\left[\sum_{2^{k-1}<|\sqrt{\lambda_n}-\mu|<2^k} \frac{|u_n(x)|^2}{|\sqrt{\lambda_n}-\mu|^2}\right] + \right. \tag{40}$$

$$\left. + \frac{4}{2^{m-1}}\cdot\|q\|_{L^1(I_R)}\cdot\int_{I_R}|q(\xi)|\sum_{n=1}^{\infty}\frac{|u_n(\xi)|^2}{\lambda_n}\,d\xi\right\}.$$

Отсюда, используя (28) и (29) получим:

$$S_2 \leq 2\sum_{k=m}^{p-1}\cdot\frac{1}{4^{k-1}}\left[\sum_{0\leq|\sqrt{\lambda_n}-\mu|\leq 2^k}|u_n(x)|^2\right] + O\left(\frac{1}{2^{m-1}}\right) = O\left(\frac{1}{2^{m-1}}\right). \tag{41}$$

Для оценки суммы, стоящей в правой части (41) в фигурных скобках, используем равномерную на каждом компакте интервала G оценку (28), положив в этой оценке $\varrho=2^k$. Отстается фиксировать номер m таким, чтобы выполнялось неравенство

$$\left|O\left(\frac{1}{2^{m-1}}\right)\right| \leq \frac{R}{8\pi^2}.$$

Сопоставляя наши оценки получим

$$S_2 < \frac{R}{8\pi^2} \tag{42}$$

и тем самым утверждение a) теоремы доказано. Доказательство утверждения b) теоремы аналогично. Теорема доказана.

Наконец переходим к доказательству следствий теоремы. Следствие 1 сразу вытекает из доказанной теоремы. Докажем Следствие 2. Интегрируя обе части формул среднего значения Э. Ч. Титчмарша

$$\frac{u_n(x+t)+u_n(x-t)}{2}=u_n(x)\cos\sqrt{\lambda_n}\,t-$$

$$-\frac{1}{2\sqrt{\lambda_n}}\cdot\int_{x-t}^{x+t}q(\xi)\,u_n(\xi)\sin\sqrt{\lambda_n}\,(|x-\xi|-t)\,d\xi\,,$$

получим

$$\int_{x-y}^{x+y}u_n(t)\,dt=u_n(x)\int_0^y\cos\sqrt{\lambda_n}\,t\,dt-\tag{43}$$

$$-\frac{1}{2\sqrt{\lambda_n}}\int_0^y\int_{x-y}^{x+y}q(\xi)\,u_n(\xi)\sin\sqrt{\lambda_n}\,(|x-\xi|-t)\,d\xi\,dt\,,$$

$$(x-y,\ x+y\in G)\,.$$

Фиксируем произвольную точку x интервала G. Выберем такую последовательность $\{n_n\}$ натуральных чисел, для которых выполняется (12). Существование такой последовательности гарантируется следствием I. После этого выберем такое число y из интервала $(0, \varrho(x, \partial G))$, для которого найдется подпоследовательность $\{n_{k_i}\}$ последовательности $\{n_k\}$ обладающая свойством:

$$\left|\int_0^y\cos\sqrt{\lambda_{n_{k_i}}}\,t\,dt\right|\geq\frac{c}{\sqrt{\lambda_{n_{k_i}}}}>0,\quad(i=1,2,\ldots)\,.\tag{44}$$

Докажем, что такой выбор y возможен из любого интервала $(0, \varepsilon)$, $(\varepsilon>0)$. Предположим противное т. е. что найдется $\varepsilon_0>0$, такое что не существует $y\in(0, \varepsilon_0)$ обладающее выше свойствами. Отсюда получаем, что

$$\sin\sqrt{\lambda_{n_k}}\;y\to 0,\quad k\to\infty,\quad 0<y<\varepsilon_0\,.$$

По теореме Лебега о предельном переходе под знаком интеграла вытекает:

$$\lim_{k\to\infty}\int_0^{\varepsilon_0}\sin^2\sqrt{\lambda_{n_k}}\;y\,dy=0\,.\tag{45}$$

Сделаем замену переменной под интегралом, имеем:

$$\lim_{k\to\infty}\frac{1}{\sqrt{\lambda_{n_k}}}\int_0^{\varepsilon_0\sqrt{\lambda_{n_k}}}\sin^2\xi\,d\xi=0\,.\tag{46}$$

С другой стороны, легко видеть, что

$$\int_0^{\varepsilon_0\sqrt{\lambda_{n_k}}}\sin^2\xi\,d\xi>c\sqrt{\lambda_{n_k}}$$

а это противоречит (46) и оценка (44) доказана.

Замечание 1. Оценка (44) вытекает из общей теоремы Дирихле: для любой фиксированной последовательность вещественных чисел $\{a_n \to \infty\}$ $(n \to \infty)$ множество $\{e^{ia_n x}\}_{n=1}^{\infty}$ плотное на единичном круге комплексной плоскости при почти каждый вещественный x.

Для завершения доказательства Следствий 2 остается показать что для уже выбранных x и y, второе слагаемое в правой части (43) имеет порядок $o(1/\sqrt{\lambda_n})$ при $n \to \infty$. Исходим из тождества работы [6]:

$$\int_0^y h_n(x, t)\, dt \overset{\text{опр}}{=} \frac{1}{2\sqrt{\lambda_n}} \int_0^y \int_{x-t}^{x+t} q(\xi)\, u_n(\xi) \sin \sqrt{\lambda_n}\, (|x-\xi| - t)\, d\xi\, dt =$$

(47)

$$= \frac{1}{2\sqrt{\lambda_n}} \int_0^y \left\{ \int_0^t [q(x+\tau)\, u_n(x+\tau) - q(x-\tau)\, u_n(x-\tau)]\, d\tau \right\} \sin \sqrt{\lambda_n}\, (t-y)\, dt.$$

Мы использовали это равенство в работе [6] без доказательства и поэтому мы дадим здесь доказательство для его.

Используя тождество

(48) $$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

получим:

$$\int_0^y h_n(x, t)\, dt = \frac{1}{2\sqrt{\lambda_n}} \int_0^y \cos \sqrt{\lambda_n}\, t \left[\int_{x-t}^{x+t} q(\xi)\, u_n(\xi) \sin \sqrt{\lambda_n}\, |x-\xi|\, d\xi \right] dt -$$

(49) $$- \frac{1}{2\sqrt{\lambda_n}} \int_0^y \sin \sqrt{\lambda_n}\, t \left[\int_{x-t}^{x+t} q(\xi)\, u_n(\xi) \cos \sqrt{\lambda_n}\, |x-\xi|\, d\xi \right] dt.$$

Из последнего равенства интегрируя по частям вытекает:

$$v_n \overset{\text{опр}}{=} \int_0^y h_n(x, t)\, dt = \frac{1}{2\sqrt{\lambda_n}} \int_0^y \left(\int_t^y \cos \sqrt{\lambda_n}\, \tau\, d\tau \right) \times$$

$$\times [q(x+t)\, u_n(x+t) - q(x-t)\, u_n(x-t)] \sin \sqrt{\lambda_n}\, t\, dt -$$

$$- \frac{1}{2\sqrt{\lambda_n}} \int_0^y \left(\int_t^y \sin \sqrt{\lambda_n}\, \tau\, d\tau \right) \times$$

$$\times [q(x+t)\, u_n(x+t) - q(x-t)\, u_n(x-t)] \cos \sqrt{\lambda_n}\, t\, dt =$$

$$= \frac{1}{2\lambda_n} \int_0^y (\sin \sqrt{\lambda_n}\, y - \sin \sqrt{\lambda_n}\, t)\, [q(x+t)\, u_n(x+t) - q(x-t)\, u_n(x-t)]\, dt +$$

$$+\frac{1}{2\lambda_n}\int_0^y \left(\cos\sqrt{\lambda_n}\,y-\cos\sqrt{\lambda_n}\,t\right)\left[q(x+t)\,u_n(x+t)-q(x-t)\,u_n(x-t)\right]dt=$$

$$=\frac{1}{2\lambda_n}\int_0^y \left[\cos\sqrt{\lambda_n}\,(y-t)-1\right]\cdot\left[q(x+t)\,u_n(x+t)-q(x-t)\,u_n(x-t)\right]dt\,.$$

Отсюда опять интегрируя по частьям вытекает тождество (47). Из (47) используя неравенство Коши—Буняковского получим:

$$|v_n|^2\le\frac{y}{4\lambda_n}\int_0^y\left|\int_0^t\left[q(x+\tau)\,u_n(x+\tau)-q(x-\tau)\,u_n(x-\tau)\right]d\tau\right|^2dt\le$$

(50)

$$\le\frac{2y}{4\lambda_n}\int_0^y\left\{\left|\int_0^t q(x+\tau)\,u_n(x+\tau)\,d\tau\right|^2+\left|\int_0^t q(x-\tau)\,u_n(x-\tau)\,d\tau\right|^2\right\}dt\,.$$

По теореме Римана—Лебега:

$$\left.\begin{aligned}\lim_{n\to\infty}\int_0^t q(x+\tau)\,u_n(x+\tau)\,d\tau=0\\ \lim_{n\to\infty}\int_0^t q(x-\tau)\,u_n(x-\tau)\,d\tau=0\end{aligned}\right\}0\le t\le y.$$

делее используя неравенство Гёлдера легко убедиться в законности предельного перехода под знаком интеграла правой части (50). Таким образом оценка $v_n=o\left(1/\sqrt{\lambda_n}\right)$ и тем самым Следствие 2 доказано.

Надо ещё доказать, что постоянную $C(x)$ в оценке (13) можно выбрать одним и тем же на каждом компакте точек x интервала G. Для этого достаточно доказать оценку $v_n=o\left(1/\sqrt{\lambda_n}\right)$ равномерно на любом компакте интервала G. Эта оценка вытекает из неравенства (50) применяя следующий факт, имеющий самостоятельный интерес:

Пусть $\{\varphi_n\}_1^\infty$ произвольная полная ортонормированная система на конечном или бесконечном интервале G, пусть $f\in L^2(G)$ любая функция и $a\in G$ любая точка. Тогда последовательность непрерывных функций

$$\int_a^x f(t)\,\varphi_n(t)\,dt;\quad n=1,2,\,\ldots$$

стремится к нулю равномерно на любом компакте интервала G. Это вытекает из равенство Парсеваля:

(51)
$$\sum_{n=1}^\infty\left|\int_a^x f(t)\,\varphi_n(t)\,dt\right|^2=\int_a^x|f(t)|^2\,dt$$

применяя известную теорему Дини (см. например В. А. Ильин – Э. Г. Позняк [8]). По этой теореме ряд непрерывных функций в равенстве (51) сходится равномерно по x на любом отрезке K интервала G, следовательно члены ряда стремятся к нулю равномерно по x на любом компакте интервала G. Отсюда сразу вытекает: если $f \in L^1(G)$ и система $\{\varphi_n\}$ равномерно ограничена на замкнутом множестве супп f (носитель функции f), то последовательность непрерывных функций (51) сходится к нулю равномерно по x на любом компакте интервала G.

Замечание 2. Известно, что если последовательность функционалов на Банаховом пространстве стремится к непрерывному функционалу в каждой точке, то эта сходимость равномерна на любом компакте этого пространства ([10]). Из этого вытекают наши утверждения, используя известную характеризацию компактных множеств в пространствап L^p.

Доказательство. Следствий 3 аналогично к доказательству Следствий 2.

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AN ELLIPTIC BOUNDARY VALUE PROBLEM FOR NONLINEAR EQUATIONS IN UNBOUNDED DOMAINS

By

F. H. MICHAEL

II. Department of Analysis of the L. Eötvös University, Budapest

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The purpose of this paper is to deal with the existence and the uniqueness of variational solution for the equation

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), \dots, D^\beta u(x), \dots)+$$
$$+g_1(x, u(x))+g_2(x, u(x), \dots, D^\gamma u(x), \dots) = f(x),$$

$x\in\Omega$, where Ω is a (possibly unbounded) domain in $\mathbf{R}^n$, under some conditions on $A_\alpha, g_1, g_2, \Omega$; $|\beta|\le m$ and $|\gamma|<m$. The main result is stated in Theorem 1. Problem of this kind has been well studied by J. R. L. WEBB in [3]. The conditions on A_α in the present paper are more general than those in [3] and the problem in [3] is discussed without the term $g_2(x, u(x), \dots, D^\gamma u(x), \dots)$.

FORMULATION OF THE PROBLEM. We shall discuss the existence and the uniqueness of variational solution for the equation

(1) $$L(u)\equiv A(u(x))+g_1(x, u(x))+$$
$$+g_2(x, u(x), \dots, D^\gamma u(x), \dots) = f(x),$$

$x\in\Omega$, where Ω is an arbitrary (may be unbounded) domain in $\mathbf{R}^n$ and A is a nonlinear elliptic partial differential operator of order $2m$ given in the form

(2) $$A(u(x)) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), \dots, D^\beta u(x), \dots),$$

$|\beta|\le m$, $0\le|\gamma|\le l$, where l is an integer such that $l<m$. More exactly we seek for a solution $u\in W^m_{p,0}(\Omega)$ satisfying the condition

$$\sum_{|\alpha|\le m} \int_\Omega A_\alpha(x, u(x), \dots, D^\beta u(x), \dots) D^\alpha \bar v\, dx+\int_\Omega g_1(x, u(x))\bar v\, dx+$$
$$+\int_\Omega g_2(x, u(x), \dots, D^\gamma u(x), \dots)\bar v\, dx = (f, \bar v)$$

for any $v\in W^m_{p,0}(\Omega)\cap L^\infty(\Omega)$ and for $v=u$, where $1<p<\infty$.

Here we use the notation (f, u) for the value of $f \in (W^m_{p,0}(\Omega))^*$ at $u \in W^m_{p,0}(\Omega)$. The expression $W^m_{p,0}(\Omega)$ will denote the completion of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{W^m_p(\Omega)}$, where

$$\|u\|_{W^m_{p,0}(\Omega)} = \|u\|_{W^m_p(\Omega)} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)}\Big)^{1/p},$$

and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

in the sense of distributions.

Assumptions on A. The operator A given by (2) is restricted as follows: $A_\alpha(x, \xi): \Omega \times \mathbf{R}^N \to \mathbf{R}$ satisfy the Carathéodory conditions, i.e., are measurable in x for each fixed $\xi \in \mathbf{R}^N$ and continuous in ξ for almost all x in Ω. N is the number of multi-indices α with $|\alpha| \leq m$, $\xi = (\xi^0, \dots, \xi^\beta, \dots)$, $|\beta| \leq m$.

(A1) There exists a constant $c \geq 0$ such that

(3) $$|A_\alpha(x, \xi)| \leq c\,(|\xi|^{p-1} + K(x, \xi)), \quad \text{for all} \quad |\alpha| \leq m,$$

where $K(x, \xi)$ is a measurable function such that

(4) $$\|u\|^p_{W^m_{p,0}} \leq c_1 \quad \text{implies} \quad \int_\Omega [K(x, u(x), \dots, D^\beta u(x), \dots)]^q\, dx \leq c_2,$$

c_1, c_2 are constants and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

(A2) There exist constants $c_3 > 0$, $c_4 > 0$ such that for all $u \in W^m_{p,0}(\Omega)$

(5) $$\sum_{|\alpha| \leq m} \operatorname{Re} \langle A_\alpha(x, u(x), \dots, D^\beta u(x), \dots), D^\alpha u \rangle \geq$$

$$\geq c_3 \|u\|^p_{W^m_{p,0}(\Omega)} - c_4,$$

where

$$\langle A_\alpha(x, u(x), \dots, D^\beta u(x), \dots), D^\alpha v \rangle =$$

$$= \int_\Omega A_\alpha(x, u(x), \dots, D^\beta u(x), \dots) \cdot D^\alpha \bar{v}\, dx.$$

(A3) Condition of semibounded variation of A: There exist a function $\Psi \in C_0^\infty(\Omega)$ and a function $(R, \varrho) \to F(R, \varrho)$ which is continuous for all fixed $R > 0$, $\lim_{\xi \to 0} \dfrac{F(R, \xi \varrho)}{\xi} = 0$ such that for any $R > 0$ and all $u, v \in W^m_{p,0}(\Omega)$ satisfying $\|u\|_{W^m_{p,0}(\Omega)} \leq R$, $\|v\|_{W^m_{p,0}(\Omega)} \leq R$ it is true that

(6) $$\operatorname{Re}(A(u) - A(v), u - v) \geq -F\Big(R, \|(u - v)\Psi\|_{W^{m-1}_{p,0}(\Omega)}\Big).$$

In [4] there is an example where (A1)–(A3) are fulfilled.

ASSUMPTIONS ON g_1, g_2.

(G1) For all $x\in\Omega$, $\xi' = (\xi^0, \dots, \xi^\gamma, \dots)\in\mathbf{K}^{N'}$, $|\gamma| \geq l$, $t\in\mathbf{K}$

$$g_1(x,t) = p_1(x,t) + r_1(x,t), \tag{7}$$

$$g_2(x,\xi') = p_2(x,\xi') + r_2(x,\xi'), \tag{8}$$

where g_1, g_2, p_1, p_2, r_1, r_2 satisfy the Carathéodory conditions, i.e., are measurable in x for each $t\in\mathbf{K}$, $\xi'\in\mathbf{K}^{N'}$ and continuous in t, ξ' for almost all x in Ω and

$$p_1(x,t)\bar{t}\geq 0, \quad |r_1(x,t)|\leq h_1(x), \quad h_1, h_2\in L^q(\Omega)\cap L^1(\Omega)$$

$$p_2(x,\xi')\bar{\xi}^0\geq 0, \quad |r_2(x,\xi')|\leq h_2(x),$$

for all $t\in\mathbf{K}$, $\xi'\in\mathbf{K}^{N'}$, $\xi^0\in\mathbf{K}$, $x\in\Omega$. Here $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$.

(G2) The equations

$$g_{1s}^*(x) = \sup_{|t|\leq s} |g_1(x,t)|, \quad g_{2s}^*(x) = \sup_{|\xi'|\leq s} |g_2(x,\xi')| \tag{9}$$

define $L^1(\Omega)$ functions for $0\leq s<\infty$.

(G3) $\|u\|_{W^m_{p,0}(\Omega)}\leq c_1$ implies

$$\int_\Omega |g_2(x, u(x), \dots, D^\gamma u(x), \dots)\cdot D^\alpha u(x)|\,dx\leq c_2, \quad |\alpha|\leq l. \tag{10}$$

REMARK. Let Ω be a region in n-dimensional Euclidean space with sufficiently smooth boundary. From the imbedding theorems (see [7]), if $l\leq m-\dfrac{n}{p}$, then the imbedding $W^m_p(\Omega)\subset C^l_*(\overline{\Omega})$ is bounded and in this case the condition (G3) is fulfilled if (G1), (G2) are satisfied. For $l>m-\dfrac{n}{p}$, from the imbedding theorems, too, the imbedding $W^m_p(\Omega)\subset W^l_{q^*}(\Omega)$ is bounded, where

$$q^* < q = \frac{np}{n-(m-l)p}$$

and in this case instead of (G3) we must have the following condition:

$$\|u\|_{W^m_{p,0}(\Omega)}\leq c_1$$

implies

$$\int_\Omega |g_2(x, u(x), \dots, D^\gamma u(x), \dots)|^{p^*}\,dx\leq c_2,$$

where $\dfrac{1}{p^*}+\dfrac{1}{q^*}=1$.

Here $C_*^l(\overline{\Omega})$ is defined by

$$C_*^l(\overline{\Omega}) = \left\{g \in C^l(\overline{\Omega});\quad \sup_{\Omega} |D^\beta g| < \infty,\quad 0 \leq |\beta| \leq l\right\}$$

and $C^l(\overline{\Omega})$ is the space of functions which belong to $C^l(\Omega)$ and the derivatives up to order l have continuous extension on $\overline{\Omega}$, the closure of Ω.

Definition. For a compact set K, the (m, p) capacity is defined by

$$C_{m,p}(K) = \inf\{\|\Phi\|_{m,p}^p;\ \Phi \in C^\infty(\mathbf{R}^n),\ \Phi \geq 1 \text{ on } K\},$$

$\|\cdot\|_{m,p}$ denotes a norm on $W_p^m(\mathbf{R}^n)$.

The definition is extended to arbitrary set E by

$$C_{m,p}(E) = \sup\{C_{m,p}(K);\ K \subset E,\ K \text{ compact}\}.$$

A statement that holds except on a set of (m, p) capacity zero is said to hold (m, p) q.e. (quasi-everywhere).

Assumption on Ω.

(Ω1) Given $u \in W_p^m(\mathbf{R}^n)$, the necessary and sufficient condition that $u \in W_{p,0}^m(\Omega)$ is $D^\alpha u = 0$ for $(m - |\alpha|, p)$ q.e. x in $C\Omega$, the complement of Ω, $|\alpha| \leq m-1$.

Remark. If $m > \max\left(\frac{n}{2},\ 2 - \frac{1}{n}\right)$, then ($\Omega$1) is fulfilled for any domain. (See [5]).

Now we shall mention an example of a problem which satisfies the assumptions (A1)–(A3), (G1)–(G3).

Example. Consider the equation

$$(-1)^m \sum_{|\alpha|=m} D^\alpha \Big(|D^\alpha u|^{p-2} D^\alpha u + \sum_{|\beta| \leq m-1} f_{\alpha\beta}(x)\, |D^\beta u|^{\varrho-1} D^\alpha u\Big) +$$

$$+ \sum_{|\beta| \leq m-1} (-1)^{|\beta|} D^\beta \Big(\prod_{|\omega| \leq m} a_{\beta\omega}(x)\, |D^\omega u|^{p_{\beta\omega}}\Big) +$$

$$+ \Phi_1(x) \Psi_1(u(x)) + \Phi_2(x) \Psi_2(u(x))\, \Psi_3(u(x), \ldots, D^\gamma u(x), \ldots) = f(x),$$

where $|\gamma| \leq l$, $1 < \varrho < p-1$, $\sum_{|\omega| \leq m-1} p_{\beta\omega} \leq \varrho$, $p_{\beta\omega} \geq 0$ and $a_{\beta\omega}, f_{\alpha\beta} \in C_0(\Omega)$, the space of continuous functions which have compact support contained in Ω; Φ_1, Φ_2 are non-negative L^1-functions; Ψ_1, Ψ_2, Ψ_3 are any continuous functions such that $\Psi_3(\xi^0, \ldots, \xi^\gamma, \ldots) \geq 0$ and

$$\Psi_1(t) = \begin{cases} \geq 0, & \text{if } t \geq 0 \\ \leq 0, & \text{if } t \leq 0 \end{cases}, \qquad \Psi_2(\xi^0) = \begin{cases} \geq 0, & \text{if } \xi^0 \geq 0 \\ \leq 0, & \text{if } \xi^0 \leq 0 \end{cases}.$$

Here $r_1(x, t) = 0$, $r_2(x, \xi') = 0$.

Theorem 1. *Let assumptions* (A1)–(A3), (G1)–(G3) *and* (Ω1) *be satisfied. Then for any* $f \in (W_{p,0}^m(\Omega))^*$ *there exists* $u \in W_{p,0}^m(\Omega)$ *such that* $g_1(\cdot, u)$,

$g_2(\cdot, u, \ldots, D^\gamma u, \ldots)$, $u[g_1(\cdot, u)+g_2(\cdot, u, \ldots, D^\gamma u, \ldots)]$ *belong to* $L^1(\Omega)$ *and*

$$\sum_{|\alpha|\leq m} \int_\Omega A_\alpha(x, u, \ldots, D^\beta u, \ldots)\cdot D^\alpha \bar{v}\, dx + \tag{11}$$

$$+\int_\Omega g_1(x, u)\cdot \bar{v}\, dx + \int_\Omega g_2(x, u, \ldots, D^\gamma u, \ldots)\cdot \bar{v}\, dx = (f, \bar{v})$$

for all $v\in W^m_{p,0}(\Omega)\cap L^\infty(\Omega)$ *and for* $v=u$.

To prove this theorem we need some other theorems and lemmas. The operator $A: W^m_{p,0}(\Omega)\to(W^m_{p,0}(\Omega))^*$ is said to be semicontinuous if for any $x, y\in W^m_{p,0}(\Omega)$, the mapping

$$t\in[0, 1]\to\big(A(x-tx+ty), x-y\big)$$

is continuous.

A is said to be bounded if it maps bounded subsets of $W^m_{p,0}(\Omega)$ into bounded subsets of $(W^m_{p,0}(\Omega))^*$.

We call A pseudomonotone whenever (u_j) is a sequence in $W^m_{p,0}(\Omega)$ which converges weakly to an element u in $W^m_{p,0}(\Omega)$ (we write $u_j\rightharpoonup u$), $A(u_j)$ converges weakly in $(W^m_{p,0}(\Omega))^*$ to f and $\limsup \operatorname{Re}\big(A(u_j), u_j-u\big)\leq 0$, then $f=A(u)$ and $\operatorname{Re}\big(A(u_j), u_j-u\big)\to 0$.

The operator A is said to be coercive if

$$\lim_{\|u\|_{W^m_{p,0}(\Omega)}\to\infty} \frac{\operatorname{Re}\big(A(u), \bar{u}\big)}{\|u\|_{W^m_{p,0}(\Omega)}} = \infty .$$

We say that A is continuous in finite dimension if for all fixed $u_1, u_2, \ldots, u_k$, $v\in W^m_{p,0}(\Omega)$ and for any sequence $(c^{(j)})$ converging to $c^{(0)}$, where $c^{(j)}=(c_1^{(j)}, c_2^{(j)}, \ldots, c_k^{(j)})\in\mathbf{R}^k$ for $j=0, 1, 2, \ldots$; $\big(A(c_1^{(j)}u_1+\ldots+c_k^{(j)}u_k), v\big)\to\big(A(c_1^{(0)}u_1+c_2^{(0)}u_2+\ldots+c_k^{(0)}u_k), v\big)$ as $j\to\infty$.

THEOREM 2. *If* $A: W^m_{p,0}(\Omega)\to(W^m_{p,0}(\Omega))^*$ *is semicontinuous and satisfies the condition of semibounded variation, then A is pseudomonotone operator.*

PROOF. First, we shall prove that if $u_j\rightharpoonup u$ such that

$$\limsup \operatorname{Re}\big(A(u_j), u_j-u\big)\leq 0,$$

then

$$\operatorname{Re}\big(A(u), u-v\big)\leq \liminf \operatorname{Re}\big(A(u_j), u_j-v\big) \tag{12}$$

for all $v\in W^m_{p,0}(\Omega)$.

By assumptions:

$$\operatorname{Re}\big(A(u_j)-A(u), u_j-u\big)\geq -F\Big(R, \|(u_j-u)\Psi\|_{W^{m-1}_{p,0}(\Omega)}\Big).$$

As $u_j \to u$, then there exists a number $R > 0$ such that:

$$\|u_j\|_{W^m_{p,0}(\Omega)} \leq R, \quad \|u\|_{W^m_{p,0}(\Omega)} \leq R$$

and

$$\operatorname{Re}\big(A(u), u_j - u\big) \leq \operatorname{Re}\big(A(u_j), u_j - u\big) + F\Big(R, \|(u_j - u)\Psi\|_{W^{m-1}_{p,0}(\Omega)}\Big). \tag{13}$$

$\{u_j\}$ is bounded which implies that there exists a subsequence $\{u'_j\}$ such that

$$\lim \|(u'_j - u)\Psi\|_{W^{m-1}_{p,0}(\Omega)} = 0 \tag{14}$$

(see [4]).

Hence

$$F\Big(R, \|(u'_j - u)\Psi\|_{W^{m-1}_{p,0}(\Omega)}\Big) \to 0.$$

Since $u_j \to u$, $u'_j \to u$ and $\operatorname{Re}\big(A(u), u'_j - u\big) \to 0$, from (13), we get

$$\lim \operatorname{Re}\big(A(u'_j), u'_j - u\big) = 0. \tag{15}$$

It is also true for the original sequence. Suppose that it is not so, then there exists a subsequence $\{u''_j\}$ such that

$$\lim \operatorname{Re}\big(A(u''_j), u''_j - u\big) = -d, \quad 0 < d \leq +\infty.$$

Applying the above treatment to the sequence $\{u''_j\}$ instead of $\{u_j\}$, then there exists a subsequence $\{u'''_j\}$ of the sequence $\{u''_j\}$ such that

$$\lim \operatorname{Re}\big(A(u'''_j), u'''_j - u\big) = 0$$

which is a contradiction. Therefore,

$$\lim \operatorname{Re}\big(A(u_j), u_j - u\big) = 0. \tag{16}$$

From the assumptions, we have

$$\operatorname{Re}\big(A(u_j) - A(\omega), u_j - \omega\big) \geq -F\Big(R, \|(u_j - \omega)\Psi\|_{W^{m-1}_{p,0}(\Omega)}\Big).$$

Put $\omega = (1-t)u + tv$, $v \in W^m_{p,0}(\Omega)$, $t \in (0, 1]$, we get

$$\operatorname{Re}\big(A(u_j), u_j - u\big) + \operatorname{Re}\big(A(u_j), tu - tv\big) -$$
$$- \operatorname{Re}\big(A(u - tu + tv), u_j - u + tu - tv\big) \geq$$
$$\geq -F\Big(R, \|(u_j - u + tu - tv)\Psi\|_{W^{m-1}_{p,0}(\Omega)}\Big).$$

As $j \to \infty$, by passing to subsequence, we get

$$t \operatorname{Re}\big(A(u - tu + tv), u - v\big) - F\Big(R, \|t(u - v)\Psi\|_{W^{m-1}_{p,0}(\Omega)}\Big) \leq$$
$$\leq t \liminf \operatorname{Re}\big(A(u_j), u - v\big).$$

Divide both sides by t and let $t\to 0$, we obtain

$$\mathrm{Re}\left(A(u),\, u-v\right)\leq \liminf \mathrm{Re}\left(A(u_j),\, u-v\right) = \liminf \mathrm{Re}\left(A(u_j),\, u_j-v\right)$$

which is (12).

Now, since

$$\mathrm{Re}\left(A(u_j),\, u_j-v\right) = \mathrm{Re}\left(A(u_j),\, u_j-u\right)+\mathrm{Re}\left(A(u_j),\, u-v\right)$$

and from (16)

$$\lim \mathrm{Re}\left(A(u_j),\, u_j-u\right) = 0\,,$$

we get

$$\liminf \mathrm{Re}\left(A(u_j),\, u_j-v\right) = \liminf \mathrm{Re}\left(A(u_j),\, u-v\right) = \lim \mathrm{Re}\left(A(u_j),\, u-v\right)$$

$$\therefore \liminf \mathrm{Re}\left(A(u_j),\, u_j-v\right) = \lim \mathrm{Re}\,(f,\, u-v)\,.$$

So we have

$$\mathrm{Re}\left(A(u),\, u-v\right)\leq \mathrm{Re}\,(f,\, u-v)$$

for all $v\in W^m_{p,0}(\Omega)$.

Hence $A(u)=f$ which completes the proof of the theorem.

THEOREM 3. *Suppose that* $A: W^m_{p,0}(\Omega)\to (W^m_{p,0}(\Omega))^*$ *is coercive, bounded, continuous in finite dimension and pseudomonotone. Then the equation* $A(u) = h$ *has at least one solution.* (See [6]).

LEMMA 1. Let $\{u_j\}$ be a sequence in $W^k_{p,0}(\Omega)$ which converges weakly to u in $W^k_{p,0}(\Omega)$. Then there is a subsequence $\{u_j'\}$ of $\{u_j\}$ such that for all α with $|\alpha|\leq k-1$, $D^\alpha(u_j')\to D^\alpha u$ for almost all x in Ω. (See [2]).

If we put

$$a(u, v) = \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x, u, \ldots, D^\beta u, \ldots)\cdot D^\alpha \bar{v}\, dx\,,$$

then $a(u, v)$ is well defined for $u, v\in W^m_{p,0}(\Omega)$ and $v\mapsto a(u, v)$ is a bounded linear functional. Thus it induces a map

$$T: W^m_{p,0}\,\Omega() \to (W^m_{p,0}(\Omega))^*$$

by the rule

$$\left(T(u),\, \bar{v}\right) = a(u, v)\,.$$

(A1) implies that T is bounded and continuous in finite dimension (thus T is semicontinuous operator). (A2) implies that T is coercive. Theorem 2 implies that T is pseudomonotone operator.

The terms $g_1(x, u(x))$, $g_2(x, u(x), \ldots, D^\gamma u(x), \ldots)$ are truncated as follows: let μ be a positive integer and

$$g_{1\mu}(x, t) = \chi_\mu(x)\, p_1^{(\mu)}(x, t)+r_1(x, t)\,,$$

$$g_{2\mu}(x, \xi') = \chi_\mu(x)\, p_2^{(\mu)}(x, \xi')+r_2(x, \xi')\,,$$

where

$$p_1^{(\mu)}(x, t) = \begin{cases} p_1(x, t) & \text{if} \quad |p_1(x, t)|\leq \mu \\ \mu\, \dfrac{p_1(x, t)}{|p_1(x, t)|} & \text{otherwise} \end{cases}$$

$$p_2^{(\mu)}(x, \xi') = \begin{cases} p_2(x, \xi') & \text{if} \quad |p_2(x, \xi')| \leq \mu \\ \mu \dfrac{|p_2(x, \xi')|}{p_2(x, \xi')} & \text{otherwise} \end{cases}$$

and

$$\chi_\mu(x) = \text{char. fn} \quad \text{of} \quad \{x \in \Omega, \ |x| \leq \mu\}.$$

Then

$$b_\mu(u, v) = \int_\Omega \left[g_{1\mu}(x, u(x)) + g_{2\mu}(x, u(x), \dots, D^\gamma u(x), \dots) \right] \bar{v}(x)\, dx$$

is defined for all $u, v \in W^m_{p,0}(\Omega)$ and

$$|b_\mu(u, v)| \leq c(\mu) \|v\|_{L^p(\Omega)}.$$

Thus $v \mapsto b_\mu(u, v)$ defines an element in $(W^m_{p,0}(\Omega))^*$, say $S_\mu(u)$ by the rule

$$(S_\mu(u), \bar{v}) = b_\mu(u, v).$$

Lemma 2. If $u_j \to u$ a.e., then $S_\mu(u_j) \to S_\mu(u)$ in $L^q(\Omega)$.

Proof. We have

$$|g_{1\mu}(\cdot, u_j) + g_{2\mu}(\cdot, u_j, \dots, D^\gamma u_j, \dots) - g_{1\mu}(\cdot, u) - g_{2\mu}(\cdot, u, \dots, D^\gamma u, \dots)|^q \leq$$
$$\leq c\{|g_{1\mu}(\cdot, u_j)|^q + |g_{2\mu}(\cdot, u_j, \dots, D^\gamma u_j, \dots)|^q +$$
$$+ |g_{1\mu}(\cdot, u)|^q + |g_{2\mu}(\cdot, u, \dots, D^\gamma u, \dots)|^q\}.$$

We shall prove that each of these four terms in the right hand side of the previous inequality is less than or equal to an integrable function which does not depend on j.

$$g_{1\mu}(x, u_j(x)) = \chi_\mu(x)\, p_1^{(\mu)}(x, u_j(x)) + r_1(x, u_j(x)),$$
$$|g_{1\mu}(x, u_j(x))| \leq |\chi_\mu(x)\, p_1^{(\mu)}(x, u_j(x))| + |r_1(x, u_j(x))| \leq$$
$$\leq \mu\, \chi_\mu(x) + h_1(x),$$
$$\therefore \ |g_{1\mu}(x, u_j(x))|^q \leq c\left[\mu^q (\chi_\mu(x))^q + (h_1(x))^q\right].$$

$\mu^q(\chi_\mu(x))^q$ is bounded and equal to zero outside of a bounded domain, $h_1(x) \in L^q(\Omega)$, hence

$$|g_{1\mu}(x, u_j(x))|^q \leq G_1(x),$$

where G_1 is an integrable function which does not depend on j.

Similarly,

$$g_{2\mu}(x, u_j(x), \dots, D^\gamma u_j(x), \dots) = \chi_\mu(x)\, p_2^{(\mu)}(x, u_j(x), \dots, D^\gamma u_j(x), \dots) +$$
$$+ r_2(x, u_j(x), \dots, D^\gamma u_j(x), \dots),$$
$$|g_{2\mu}(x, u_j(x), \dots, D^\gamma u_j(x), \dots)| \leq |\chi_\mu(x)\, p_2^{(\mu)}(x, u_j(x), \dots, D^\gamma u_j(x), \dots)| +$$
$$+ |r_2(x, u_j(x), \dots, D^\gamma u_j(x), \dots)| \leq \mu\, \chi_\mu(x) + h_2(x),$$

$$\therefore \ |g_{2\mu}(x, u_j(x), \ldots, D^\gamma u_j(x), \ldots)|^q \le c\left[\mu^q(\chi_\mu(x))^q + (h_2(x))^q\right]$$

since $h_2(x) \in L^q(\mu)$, then

$$|g_{2\mu}(x, u_j(x), \ldots, D^\gamma u_j(x), \ldots)|^q \le G_2(x),$$

where G_2 is an integrable function which does not depend on j.

In the same way, it is clear that $|g_{1\mu}(x, u(x))|^q$ and $|g_{2\mu}(x, u(x), \ldots, \ldots, D^\gamma u(x), \ldots)|^q$ are less than or equal to integrable functions which do not depend on j. Hence

$$|g_{1\mu}(., u_j) + g_{2\mu}(., u_j, \ldots, D^\gamma u_j, \ldots) -$$
$$- g_{1\mu}(., u) - g_{2\mu}(., u, \ldots, D^\gamma u, \ldots)|^q \le G,$$

where G is an integrable function which does not depend on j. Since $g_{1\mu}$, $g_{2\mu}$ are continuous functions, it is easy to see that

$$|g_{1\mu}(., u_j) + g_{2\mu}(., u_j, \ldots, D^\gamma u_j, \ldots) -$$
$$- g_{1\mu}(., u) - g_{2\mu}(., u, \ldots, D^\gamma u, \ldots)|^q \to 0$$

a.e. Then by using Lebesgue's dominated convergence theorem, we get

$$\int_\Omega |g_{1\mu}(., u_j) + g_{2\mu}(., u_j, \ldots, D^\gamma u_j, \ldots) -$$
$$- g_{1\mu}(., u) - g_{2\mu}(., u, \ldots, D^\gamma u, \ldots)|^q \, dx \to 0$$

i.e. $S_\mu(u_j) \to S_\mu(u)$ in $L^q(\Omega)$.

LEMMA 3. S_μ, $T+S_\mu$ are pseudomonotone and S_μ is bounded.

PROOF. As $u_j \to u$ there is a subsequence u_j' such that $u_j' \to u$ a.e., by lemma 2 we have $S_\mu(u_j') \to S_\mu(u)$ in $L^q(\Omega)$. By Hölder's inequality we obtain

$$\mathrm{Re}\,(S_\mu(u_j'), u_j' - u) \to 0 .$$

Since $S_\mu(u_j') \to S_\mu(u)$ in $L^q(\Omega)$, $S_\mu(u_j) \to S_\mu(u)$ weakly, but if $S_\mu(u_j) \to y$ as $u_j \to u$, then $S_\mu(u) = y$.

As $\limsup \mathrm{Re}\,(S_\mu(u_j), u_j - u) \le 0$ and there exists a subsequence u_j' such that

$$\mathrm{Re}\,(S_\mu(u_j'), u_j' - u) \to 0,$$

then

$$\mathrm{Re}\,(S_\mu(u_j), u_j - u) \to 0,$$

for suppose that $\mathrm{Re}\,(S_\mu(u_j), u_j - u)$ does not tend to zero. Then there exist $\varepsilon_0 > 0$ and a subsequence $\{u_j''\}$ such that

$$\mathrm{Re}\,(S_\mu(u_j''), u_j'' - u) < -\varepsilon_0 .$$

Applying the first part of the proof for u_j'' instead of u_j, then $\{u_j''\}$ has a subsequence $\{u_j'''\}$ such that

$$\mathrm{Re}\,(S_\mu(u_j'''), u_j''' - u) \to 0$$

which is a contradiction and pseudomonotonicity of S_μ follows. Similarly can be proved that $T+S_\mu$ is pseudomonotone and S_μ is bounded.

LEMMA 4. Under the above assumtions, if $\{u_j\}$ is a sequence in $W^m_{p,0}(\Omega)$ with $u_j \rightharpoonup u$ in $W^m_{p,0}(\Omega)$ and

$$\int_\Omega [|g_{1j}(.,u_j)| + |g_{2j}(.,u_j,\ldots,D^\gamma u_j,\ldots)|]\,|u_j| \leq c_1,$$

$$\int_\Omega |g_{2j}(.,u_j,\ldots,D^\gamma u_j,\ldots)| \cdot |D^\alpha u_j| \leq c_2$$

for all $|\alpha| \leq l$ and all j, then $u[g_1(.,u)+g_2(.,u,\ldots,D^\gamma u,\ldots)] \in L^1(\Omega)$ and

$$g_{1j}(.,u_j)+g_{2j}(.,u_j,\ldots,D^\gamma u_j,\ldots) \to g_1(.,u)+g_2(.,u,\ldots,D^\gamma u,\ldots)$$

in $L^1(\Omega)$ as $j \to \infty$.

PROOF. As $u_j \rightharpoonup u$ in $W^m_{p,0}(\Omega)$ there is a subsequence, again denoted by u_j, such that $u_j \to u$ a.e. in Ω and $D^\gamma u_j \to D^\gamma u$ a.e. in Ω for $|\gamma| \leq m-1$. (See lemma 1). Condition (G1) and the definitions of g_{1j}, g_{2j} imply that

$$g_{1j}(x,u_j(x)) \to g_1(x,u(x))$$

and

$$g_{2j}(x,u_j(x),\ldots,D^\gamma u_j(x),\ldots) \to g_2(x,u(x),\ldots,D^\gamma u(x),\ldots)$$

a.e. in Ω as $j \to \infty$. By Fatou's lemma

$$\operatorname{Re}\int_\Omega [g_1(.,u)+g_2(.,u,\ldots,D^\gamma u,\ldots)]\,\bar{u} \leq$$

$$\leq \liminf \operatorname{Re}\int_\Omega [g_{1j}(.,u_j)+g_{2j}(.,u_j,\ldots,D^\gamma u_j,\ldots)]\,\bar{u}_j \leq c_1,$$

$$\operatorname{Im}\int_\Omega [g_1(.,u)+g_2(.,u,\ldots,D^\gamma u,\ldots)]\,\bar{u} \leq$$

$$\leq \liminf \operatorname{Im}\int_\Omega [g_{1j}(.,u_j)+g_{2j}(.,u_j,\ldots,D^\gamma u_j,\ldots)]\,\bar{u}_j \leq c_2.$$

Hence

$$\int_\Omega |[g_1(.,u)+g_2(.,u,\ldots,D^\gamma u,\ldots)]\,\bar{u}| < \infty,$$

i.e.

$$u[g_1(.,u)+g_2(.,u,\ldots,D^\gamma u,\ldots)] \in L^1(\Omega).$$

For any $\delta > 0$, we have

$$|g_{1j}(x,u_j)+g_{2j}(x,u_j,\ldots,D^\gamma u_j,\ldots)| \leq |g_{1j}(x,u_j)| +$$

$$+ |g_{2j}(x,u_j,\ldots,D^\gamma u_j,\ldots)| \leq \sup_{|t| \leq \delta^{-1}} |g_1(x,t)| + 2h_1(x) +$$

$$+ \delta\,|g_{1j}(x,u_j)\cdot u_j| + \sup_{|\xi'| \leq \delta^{-1}} |g_2(x,\xi')| + 2h_2(x) +$$

$$+ |g_{2j}(x,u_j,\ldots,D^\gamma u_j,\ldots)|\,\delta \sum_{|\alpha| \leq l} |D^\alpha u_j|.$$

Thus for any measurable set E in Ω,

$$\int_E |g_{1j}(\cdot, u_j)+g_{2j}(\cdot, u_j, \ldots, D^\gamma u_j, \ldots)|\, dx \le$$

$$\le \int_E [g^*_{1\delta^{-1}}(x)+g^*_{2\delta^{-1}}(x)+2h_1+2h_2]\, dx+\delta C .$$

Given $\varepsilon>0$, let $\delta = \frac{\varepsilon}{2C}$. Then for meas (E) sufficiently small

$$\int_E |g_{1j}(\cdot, u_j)+g_{2j}(\cdot, u_j, \ldots, D^\gamma u_j, \ldots)|\, dx<\varepsilon$$

and there is a set A_ε of finite measure with

$$\int_{\Omega\setminus A_\varepsilon} |g_{1j}(\cdot, u_j)+g_{2j}(\cdot, u_j, \ldots, D^\gamma u_j, \ldots)|\, dx<\varepsilon .$$

By Vitali's theorem these show that

$$g_{1j}(\cdot, u_j)+g_{2j}(\cdot, u_j, \ldots, D^\gamma u_j, \ldots)\to g_1(\cdot, u)+g_2(\cdot, u, \ldots, D^\gamma u, \ldots)$$

in $L^1(\Omega)$.

LEMMA 5. Under the above assumptions, the following inequality holds

$$\operatorname{Re}((T+S_\mu)(u), u)\ge c_1\|u\|^p_{\overset{\circ}{W}{}^m_{p,0}(\Omega)}-c_2-[\|h_1\|_{L^q(\Omega)}+\|h_2\|_{L^q(\Omega)}]\,\|u\|_{L^p(\Omega)} ,$$

where c_1, c_2 are constants and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. Furthermore, $(T+S_\mu)$ is a coercive operator.

PROOF. We have

$$b_\mu(u, u) = \int_\Omega [g_{1\mu}(x, u)+g_{2\mu}(x, u, \ldots, D^\gamma u, \ldots)]\,\bar u(x)\, dx =$$

$$= \int_\Omega \chi_\mu(x)[p_1^{(\mu)}(x, u)+p_2^{(\mu)}(x, u, \ldots, D^\gamma u, \ldots)]\,\bar u(x)\, dx+$$

$$+\int_\Omega [r_1(x, u)+r_2(x, u, \ldots, D^\gamma u, \ldots)]\,\bar u(x)\, dx .$$

But

$$\chi_\mu(x)\, p_1^{(\mu)}(x, u(x))\,\bar u(x)\ge 0 ,$$

$$\chi_\mu(x)\, p_2^{(\mu)}(x, u(x), \ldots, D^\gamma u(x), \ldots)\,\bar u(x)\ge 0 .$$

Therefore

$$\operatorname{Re} b_\mu(u, u)\ge -\left|\int_\Omega [r_1(x, u)+r_2(x, u, \ldots, D^\gamma u, \ldots)]\,\bar u\, dx\right|$$

and we have

$$|r_1(x,t)| \le h_1(x), \qquad |r_2(x,\xi')| \le h_2(x).$$

Then

$$\operatorname{Re}\left((T+S_\mu)(u), \bar{u}\right) = \operatorname{Re}\left(T(u), \bar{u}\right) + \operatorname{Re} b_\mu(u,u),$$

by using Hölder's inequality and (A2) we get:

$$\operatorname{Re}\left((T+S_\mu)(u), \bar{u}\right) \ge c_3 \|u\|^p_{W^m_{p,0}(\Omega)} -$$

$$- c_4 - \left[\|h_1\|_{L^q(\Omega)} + \|h_2\|_{L^q(\Omega)}\right] \|u\|_{L^p(\Omega)},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Hence

$$\frac{\operatorname{Re}\left((T+S_\mu)(u), \bar{u}\right)}{\|u\|_{W^m_{p,0}(\Omega)}} \ge c_3 \|u\|^{p-1}_{W^m_{p,0}(\Omega)} - \frac{c_4}{\|u\|_{W^m_{p,0}(\Omega)}} -$$

$$- \left[\|h_1\|_{L^q(\Omega)} + \|h_2\|_{L^q(\Omega)}\right] \frac{\|u\|_{L^p(\Omega)}}{\|u\|_{W^m_{p,0}(\Omega)}}.$$

As

$$\|u\|_{W^m_{p,0}(\Omega)} \to \infty, \qquad \frac{\|u\|_{L^p(\Omega)}}{\|u\|_{W^m_{p,0}(\Omega)}} \le c.$$

Then

$$\lim_{\|u\|_{W^m_{p,0}(\Omega)} \to \infty} \frac{\operatorname{Re}\left((T+S_\mu)(u), \bar{u}\right)}{\|u\|_{W^m_{p,0}(\Omega)}} = +\infty$$

i.e. $(T+S_\mu)$ is a coercive operator.

Proof of theorem 1. On the basis of the properties of T, S_j that mentioned and proved above, thus, from theorem 3, $T+S_j$ maps $W^m_{p,0}(\Omega)$ onto $(W^m_{p,0}(\Omega))^*$, in particular, there exists a $u_j \in W^m_{p,0}(\Omega)$ such that

$$T(u_j) + S_j(u_j) = f \quad \text{in} \quad (W^m_{p,0}(\Omega))^*.$$

Moreover $\|u_j\| \le c$, for if not, then from lemma 5

$$\frac{\operatorname{Re}(f, \bar{u}_j)}{\|u_j\|_{W^m_{p,0}(\Omega)}} = \frac{\operatorname{Re}\left((T+S_j)(u_j), \bar{u}_j\right)}{\|u_j\|_{W^m_{p,0}(\Omega)}} \to \infty \quad \text{as} \quad \|u_j\|_{W^m_{p,0}(\Omega)} \to \infty.$$

But we have

$$\frac{|(f, \bar{u}_j)|}{\|u_j\|_{W^m_{p,0}(\Omega)}} \le \frac{\|f\|_{L^q(\Omega)} \|u_j\|_{L^p(\Omega)}}{\|u_j\|_{W^m_{p,0}(\Omega)}}$$

which is finite as $\|u_j\|_{W^m_{p,0}(\Omega)} \to \infty$. Then we have a contradiction. As T is bounded, by passing to subsequences, we may suppose that $u_j \rightharpoonup u$ in $W^m_{p,0}(\Omega)$ and $T(u_j) \rightharpoonup y$ in $(W^m_{p,0}(\Omega))^*$. Thus $\|u_j\| \le c_1$, $\|T(u_j)\| \le c_2$ and

$$2 \int_\Omega [|r_1(.,u_j)| + |r_2(.,u_j)|]\, |u_j| \le c_3$$

imply that

$$\int_\Omega [|g_{1j}(.,u_j)| + |g_{2j}(.,u_j,\ldots,D^\gamma u_j,\ldots)|]\, |u_j| \le$$
$$\le \|f\| \cdot c_1 + c_1 c_2 + c_3 = c$$

and from (G3), we get

$$\int_\Omega |g_{2j}(.,u_j,\ldots,D^\gamma u_j,\ldots) \cdot (D^\alpha \bar{u}_j)|\, dx \le c, \quad \text{for} \quad |\alpha| < m.$$

Thus by lemma 4,

$$u\,[g_1(.,u) + g_2(.,u,\ldots,D^\gamma u,\ldots)] \in L^1(\Omega)$$

and

$$g_{1j}(.,u_j) + g_{2j}(.,u_j,\ldots,D^\gamma u_j) \to g_1(.,u) + g_2(.,u,\ldots,D^\gamma u,\ldots)$$

in $L^1(\Omega)$. Therefore for any $v \in W^m_{p,0}(\Omega) \cap L^\infty$, passing to the limit as $j \to \infty$, we obtain

$$(y,\bar{v}) + \int_\Omega [g_1(.,u) + g_2(.,u,\ldots,D^\gamma_a u,\ldots)]\, \bar{v}\, dx = (f,\bar{v}). \tag{17}$$

We shall show that $y = T(u)$ by using the pseudomonotone property of T. Now,

$$(T(u_j), u_j - u) = (T(u_j), u_j) - (T(u_j), u)$$

so

$$\limsup \operatorname{Re}(T(u_j), u_j - u) = \limsup \operatorname{Re}(f - S_j(u_j), u_j) - \operatorname{Re}(y,u) \le$$
$$\le \operatorname{Re}(f - y, u) - \liminf \operatorname{Re} \int_\Omega [g_{1j}(.,u_j) + g_{2j}(.,u_j,\ldots,D^\gamma u_j,\ldots)]\, u_j\, dx.$$

From Fatou's lemma, we get

$$\limsup \operatorname{Re}(T(u_j), u_j - u) \le \operatorname{Re}(f - y, u) -$$
$$- \operatorname{Re} \int_\Omega [g_1(.,u) + g_2(.,u,\ldots,D^\gamma u,\ldots)]\, u\, dx.$$

Thus for any $\omega \in W^m_{p,0}(\Omega) \cap L^\infty$, using (17) we get

$$\limsup \operatorname{Re}(T(u_j), u_j - u) \le \operatorname{Re}(f - y, u - \omega) +$$
$$+ \operatorname{Re} \int_\Omega [g_1(.,u) + g_2(.,u,\ldots,D^\gamma u,\ldots)](\omega - u)\, dx.$$

By (Ω1), there is a sequence $\omega_j \in W^m_{p,0}(\Omega) \cap L^\infty$ such that $\omega_j \to u$ in $W^m_{p,0}(\Omega)$ and $|\omega_j(x)| \leq c|u(x)|$ a.e. (see [3]) thus $\operatorname{Re}(f-y, u-\omega_j) \to 0$ and by passing to subsequences, we get

$$\int_\Omega [g_1(.,u)+g_2(.,u,\dots,D^\gamma u,\dots)]\,\omega_j\,dx \to$$

$$\to \int_\Omega [g_1(.,u)+g_2(.,u,\dots,D^\gamma u,\dots)]\,u\,dx$$

by dominated convergence theorem, since

$$u\,[g_1(.,u)+g_2(.,u,\dots,D^\gamma u,\dots)] \in L^1(\Omega).$$

Hence

$$\limsup \operatorname{Re}\big(T(u_j), u_j - u\big) \leq 0.$$

So that $y = T(u)$ and $\big(T(u_j), u_j - u\big) \to 0$.

From (17), then

$$a(u,v) + \int_\Omega [g_1(.,u)+g_2(.,u,\dots,D^\gamma u,\dots)]\,\bar v\,dx = (f,\bar v).$$

Setting $v = \omega_j$ and let $j \to \infty$, we obtain

$$a(u,u) + \int_\Omega [g_1(.,u)+g_2(.,u,\dots,D^\gamma u,\dots)]\,\bar u\,dx = (f,\bar u)$$

which completes the proof of the theorem.

THEOREM 4. *Suppose that the operator A defined by* (2) *satisfies* (A1) *and the following condition of strong ellipticity:*

$$\sum_{|\alpha|\leq m} \operatorname{Re}\langle A_\alpha(x,u,\dots,D^\beta u,\dots) - A_\alpha(x,v,\dots,D^\beta v,\dots), D^\alpha(u-v)\rangle \geq$$

(18)

$$\geq a_1 \|u-v\|^p_{W^m_{p,0}(\Omega)},$$

where $a_1 > 0$ is a constant. Then the problem (1) *under the conditions* (G1), (G2) *and* (Ω1) *has one and only one solution provided that $g_2(x,\xi') = 0$ and $g_1(x,t)$ satisfies the condition that:*

$$g_1(x,t_1) \geq g_1(x,t_2) \quad \textit{for} \quad t_1 \geq t_2.$$

REMARK. If A statisfies the following condition

$$\sum_{|\alpha|\leq m} [A_\alpha(x,\xi) - A_\alpha(x,\eta)]\,(\bar\xi_\alpha - \bar\eta_\alpha) \geq c_1 \sum_{|\alpha|\leq m} |\xi_\alpha - \eta_\alpha|^p,$$

where $c_1 > 0$ is a constant, then A satisfies (18).

A is called a strictly monotone operator if

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \geq 0, \quad \text{for any} \quad u_1, u_2 \in D(A),$$

and

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle = 0 \quad \text{iff} \quad u_1 \equiv u_2 ,$$

where $D(A)$ is the domain of definition of A.

PROOF OF THEOREM 4. From (18) it follows that condtions (A2) and (A3) are fulfilled. The conditions of the theorem imply that $L(u)$ defined by (I) is strictly monotone, therefore the solution is unique.

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ON ISOMETRIC AND ISOMORPHIC CHARACTERIZATIONS OF SPACES OF CONTINUOUS FUNCTIONS

By

A. BÁRDOSSY

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1. Introduction

The isometric and isomorphic theory of $C(K)$ spaces (K compact Hausdorff space) is quite developped. There are a lot of conditions which guarantee an E Banach space to be isometric to a $C(K)$ space. ([1], [4], [7]). In this paper we first present another isometric characterization. The similar isomorphic characterization is also given. Finally the results are applied to M spaces.

2. Isometric case

First we introduce the condition (A)

DEFINITION 1. The Banach space E satisfies the condition (A) if there is an $e \in E$ such that for all $x \in E$ we have

$$\max(\|x+e\|, \|x-e\|) = \|x\|+1 .$$

Let $C = \{x; \|2x - \|x\| e\| = \|x\|\}$.

LEMMA 1. C is a positive cone.

PROOF. From the condition (A) $\|e\| = 1$.

If $a > 0$ the $aC \subset C$.

If $x \neq 0$ then (A) implies

$$\max(\|2x - \|x\| e\|, \|-2x - \|x\| e\|) = 3\|x\|$$

so $C \cap (-C) = \Phi$.

We only have to prove that $C + C \subset C$. Let

$$Z = \{x^* \in E^*, \|x^*\| = 1, |x^*(e)| = 1\}.$$

We shall prove that

$$\|x\| = \sup\{|x^*(x)|;\ x^* \in Z\}. \tag{1}$$

We can assume that $\|x+e\| = \|x\|+1$. Using the Hahn—Banach theorem, we can find an $x^*\in E^*$ with $\|x^*\| = 1$ such that:

$$\|x\|+1 = |x^*(x+e)| \leq |x^*(x)| + |x^*(e)| \leq \|x\|+1$$

so $|x^*(e)| = 1$ and (1) is proved. Let

$$C_1 = \left\{x;\ \frac{x^*(x)}{x^*(e)} \geq 0; \quad \forall x^* \in Z\right\}.$$

For this C_1 we have $C_1 = C$ because if $x\in C_1$,

$$\|2x-\|x\|e\| = \sup\{|2x^*(x)-\|x\|x^*(e)|,\ x^*\in Z\} = \|x\|$$

and if $x\in C$ then by (1) we have again:

$$\sup\{|2x^*(x)-\|x\|x^*(e)|;\ x^*\in Z\} = \|x\| \quad \text{so} \quad C_1 = C.$$

For C_1 the condition $C_1+C_1\subset C_1$ is evident.

This cone C introduces a partial ordering on E. This means we have $x\geq y$ if $(x-y)\in C$.

LEMMA 2. The element e is a strong order unit in (E, C) and the norm on E is the strong order unit norm introduced by e.

PROOF. We use the set Z from lemma 1. If $\|x\|\leq 1$ we have for all $x^*\in Z$

$$\frac{x^*(e-x)}{x^*(e)} = \frac{x^*(e)-x^*(x)}{x^*(e)} \geq 0$$

and

$$\frac{x^*(e+x)}{x^*(e)} = \frac{x^*(e)+x^*(x)}{x^*(e)} \geq 0$$

so $-e\leq x\leq e$. If $\|x\|>1$ then there is an $y^*\in Z$ such that

$$|y^*(x)| > 1$$

so

$$\frac{y^*(e)-y^*(x)}{y^*(e)} \leq 0$$

or

$$\frac{y^*(e)+y^*(x)}{y^*(e)} \leq 0.$$

Which means $e-x\leq 0$ or $e+x\leq 0$. So we have $\|x\|\leq 1$ if and only if $-e\leq \leq x\leq e$.

Now we can give the isometric characterization:

THEOREM 1. *The Banach space E is linear isometric to a $C(T)$ space for some T compact Hausdorff space, if and only if E satisfies the condition* (A) *and for each $z\in E$ there is an $y\in E$ such that*

(2) $$C\subset(C+z) = C+y.$$

PROOF. If E is isometric to $C(T)$, then with $e=1$ E satisfies the condition (A). The cone C introduces the natural ordering on $C(T)$, and as $C(T)$ is a lattice it obviously satisfies (2).

Conversely if (2) holds then E is a Banach lattice. By Lemma 2 E has a strong order unit, and a strong order unit norm, so E is an abstract M space [6]. Now by KAKUTANI's theorem [5] E is lineary isometric to $C(T)$ for some compact Hausdorff space T.

3. Isomorphic case

Now a similar sufficient condition can be given for the isomorphic case. For this we introduce the condition (B).

DEFINITION 2. The Banach space E satisfies the condition (B) if there is an $f \in E$ and an $\alpha > 0$ such that for all $x \in E$

$$\max(\|x+f\|, \|x-f\|) \geq \|x\| + \alpha .$$

Now we want to find a connection between the conditions (A) and (B). For this we prove the following:

THEOREM 2. *If E satisfies the condition* (B) *then there is an equivalent norm on E with which it satisfies the condition* (A).

PROOF. Using the condition (B), for each $x \in E$ we have an $x^* \in E^*$ $\|x^*\| \leq 1$ such that

$$|x^*(x)| = \|x\|, \qquad |x^*(f)| > \alpha .$$

So let

$$H = \{x^*, |x^*(f)| \geq \alpha\} .$$

Then with the help of a subset H_1 of H for which

$$\|x\| = \sup\{|x^*(x)|;\ x^* \in H_1\}$$

holds, we can define a new norm

$$|||x||| = \sup\left\{\left|\frac{x^*(x)}{x^*(f)}\right|;\ x^* \in H_1\right\}.$$

It is clear that this is a norm on E, and we have

$$\frac{1}{\|f\|}\|x\| \leq |||x||| \leq \frac{1}{\alpha}\|x\|$$

and

$$\max\{|||x+f|||, |||x-f|||\} =$$

$$= \max\left\{\sup\left\{\left|\frac{x^*(x)+x^*(f)}{x^*(f)}\right|;\ x^* \in H_1\right\}, \sup\left\{\left|\frac{x^*(x)-x^*(f)}{x^*(f)}\right|;\ x^* \in H_1\right\}\right\} =$$

$$= |||x||| + 1 .$$

It is obvious that the cone

$$C_2 = \left\{x;\ \frac{x^*(x)}{x^*(f)} \geq 0;\ x^* \in H_1\right\}$$

coincides with the cone C defined with the norm $|||\ |||$.

So we can apply theorem I. and we obtain:

COROLLARY 1. If the Banach space E satisfies (B) and there is an H_1 which defines the cone C_2 such that E with the partial ordering introduced by C_2 is a lattice, then E is isomorphic to a $C(T)$ space.

4. Application to M spaces

Let us recall some definitions. A closed subspace X of $C(K)$ is called a G space if there exist $x_i, y_i \in K$ and numbers λ_i such that

$$X = \{f \in C(K) : f(x_i) = \lambda_i f(y_i)\ \ \forall_i \in I\}.$$

A subspace is an M space if all λ_i-s are non-negative.

KAKUTANI [5] introduced the M spaces, and proved that they coincide with the closed sublattices of $C(K)$.

BENYAMINI [2] proved that separable G spaces are isomorphic to $C(K)$ spaces. In [3] he gave an example for a non-separable M space which is not isomorphic to a $C(K)$ space. Here we give a simple sufficient condition for M spaces to be isomorphic to $C(K)$ spaces.

THEOREM 3. *If E is an M space in $C(K)$ with the set of relations*

$$\{x_i, y_i; \lambda_i\}_{i \in I} \quad \text{and} \quad 0 \notin \{\lambda_i;\ i \in I\}$$

then E is isomorphic to a $C(T)$ space for some compact Hausdorff space T.

PROOF. For each $x \in K$ we have a function $f_x \in E$ such that

$$f_x(x) > \varepsilon > 0.$$

Let $V_x = \left\{x';\ f_x(x') > \frac{\varepsilon}{2}\right\}$. Using the compactness of K $K = \bigcup_1^n V_{x_i}$. Let $\Phi = \max\{f_i,\ i = 1, \ldots, n\}$.

So $\Phi \in E$ and $\Phi(x) > \frac{\varepsilon}{2}$ for all $x \in K$. E with Φ satisfies condition (B). Let S be the set of Dirac measures on K.

$$S = \{\delta_t,\ \delta_t(f) = f(t)\}.$$

For every $g \in E$ we have $\|g\| = \sup\{|x^*(g)|;\ x^* \in S\}$ and

$$\sup\{x^*(\Phi),\ x^* \in S\} \geq \inf \Phi > 0.$$

The ordering defined by S coincides with the original ordering of E, so by Corollary 1 we have $E \sim C(T)$ for some T.

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A FUNCTIONAL EQUATION WITH POLYNOMIALS

By

J. FEHÉR

Teacher's Training College, Pécs

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1. J. BINZ [1] stated the following problem. Determine all polynomials f with complex coefficients satisfying the equation

$$f(x)f(-x) = f(x^2).$$

Let $\varepsilon_k = e^{2\pi i k/s}$ denote the s'th unit roots ($k = 1, 2, \ldots, s$). Let us consider the equation

$$\prod_{k=1}^{s} f(\varepsilon_k x) = f(x^s), \tag{1.1}$$

where f is a polynomial. In this paper we shall give all the solutions of (1.1), if the coeffitients are complex, real or rational numbers.

Let H and K denote some generalized sets of complex numbers, allowing the occurrence of their elements with multiplicity. We shall say that H is equal to K, e.g. $H \doteq K$, if they contain their elements with the same multiplicity. We shall say that a set H is a system if its elements are considered with their multiplicity.

We shall say that a system M is the union of H and K if M contains exactly those elements, that belong to H or K with the sum of the multiplicity of their occurrence in H and K. We shall use the notation: $M \doteq H \uplus K$.

In what follows let $s > 1$ be fixed.

DEFINITION 1. We shall say that α has a suitable exponent if there exists a positive integer k for which α is a root of

$$x^{s^k} - x = 0. \tag{1.2}$$

We shall say that α belongs to k, if k is the minimal integer for which α is a root of (1.2).

LEMMA 1. If α belongs to k_0, and k is a suitable exponent to α, then $k_0 | k$.

PROOF. It is obvious that $k \geqq k_0$. Let $k = mk_0 + r$, $0 \leqq r < k_0$, $m > 0$. Then

$$\alpha = \alpha^{s^{mk_0+r}} = (\alpha^{s^{mk_0}})^{s^r} = \alpha^{s^r},$$

that is possible only if $r = 0$. ▌

LEMMA 2. If α belongs to k and $k > 1$, then the complex numbers α, $\alpha^s, \ldots, \alpha^{s^{k-1}}$ are disjoint, and each of them belongs to k.

PROOF. First we prove that each of them belongs to k. It is enough to prove that α^s belongs to k.

Since

$$(\alpha^s)^{s^k} = (\alpha^{s^k})^s = \alpha^s,$$

therefore α^s has a suitable exponent, and it belongs to $h \leqq k$. If $h < k$, then

$$(\alpha^s)^{s^h} = \alpha^s,$$

after powering it to the exponent s^{k-1} we have

$$\alpha^{s^{h+k}} = (\alpha^s)^{s^h \cdot s^{k-1}} = \alpha^{s^k} = \alpha,$$

whence by Lemma 1 $k | h + k$, that involves that $h = 0$. If $\alpha^{s^i} = \alpha^{s^j}$ for $0 \leqq i < j \leqq k-1$, then

$$\alpha^{s^i} = (\alpha^{s^i})^{s^{j-i}},$$

which is impossible from $0 < j - i < k$. ▌

LEMMA 3. If α is an element of H, and

$$\text{(1.3)} \qquad H := \{\alpha_1, \ldots, \alpha_n\} \doteq \{\alpha_1^s, \ldots, \alpha_n^s\},$$

then α has a suitable exponent. If α belongs to k, then $k \leqq n$.

PROOF. 1. First we prove that, if α has a suitable exponent, then α belongs to k with $k \leqq n$. Indeed, if α belongs to k, then from $\alpha \in H$, $\alpha^s \in H, \ldots$; furthermore from Lemma 2, $\alpha, \alpha^s, \ldots, \alpha^{s^{k-1}}$ are disjoint, consequently $k \leqq n$.

2. Now we prove that H has at least one element that has a suitable exponent. This is obvious if $n = 1$. Let $n > 1$; $\alpha \in H$. Since $\alpha^s, \alpha^{s^2}, \ldots, \alpha^{s^{n+1}} \in H$, therefore for suitable i and j with $1 \leqq i < j \leqq n+1$ we have $\alpha^{s^i} = \alpha^{s^j} = (\alpha^{s^i})^{s^{j-i}}$. This involves the assertion.

3. Now we prove that each element of H has a suitable exponent. We shall use induction by n. The assertion has been proved for $n = 1$. Let $n > 1$, and assume that this is proved for every H that contains m elements, where $1 \leqq m < n$. Assume that α_1 belongs to k. From 1. we have $k \leqq n$. If $k = n$, then by using Lemma 2., we are ready. Assume that $k < n$. Since $\alpha_1^{s^i} \in H$ for every $i \geqq 1$, therefore we may assume that

$$\alpha_2 = \alpha_1^s, \quad \alpha_3 = \alpha_1^{s^2}, \ldots, \alpha_k = \alpha_1^{s^k}.$$

Let us consider the systems

$$H_1 := \{\alpha_1, \alpha_2, \ldots, \alpha_k\}; \quad H_2 := \{\alpha_{k+1}, \ldots, \alpha_n\}.$$

So $H \doteq H_1 \uplus H_2$. For the systems H, H_1 the relation (1.3) is valid, so it is true for H_2, too. Each element of H_1 belongs to k (see Lemma 2), each element of H_2 has a suitable exponent by the induction hypothesis. ∎

DEFINITION 2. The system $H = \{\alpha_1, \ldots, \alpha_n\}$ is called to be connected if it satisfies (1.3), and has at least one element that belongs to k.

LEMMA 4. If $H = \{\alpha_1, \ldots, \alpha_n\}$ is connected, then

(a) its elements are disjoint,
(b) every element belongs to n,
(c) $H = \{\alpha_i^s, \alpha_i^{s^2}, \ldots, \alpha_i^{s^n}\}$ for every $i = 1, \ldots, n$.

PROOF. It is obvious. ∎

COROLLARY 1. If the connected systems H_1 and H_2 have a common element, then $H_1 \doteq H_2$.

Lemma 5. If for H (1.3) holds, then H can be given as the union of connected subsystems. The components are uniquely determined.

PROOF. Let $H = \{\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$.
First we prove the existence. If $n = 1$, then H is connected, and we are ready. Let $n > 1$, and assume that the existence of splitting is proved for every system satisfying (1.3) with $m < n$ elements. Assume that H is not connected. Assume that α_1 belongs to k. By a suitable rearrangement of the elements we may assume that $H_1 = \{\alpha_1, \ldots, \alpha_k\}$ is a connected system. Let $H_2 = \{\alpha_{k+1}, \ldots, \alpha_n\}$. It is obvious that (1.3) holds for H_2. Since $n - k < n$, we can use the induction hypothesis for H_2.
Now we prove the unicity. Let $\alpha \in H$ with multiplicity m. Let us consider a splitting. Then α belongs to some of the components. Let $\alpha \in H_1$. Then H_1 can be represented by α. H_1 contains α exactly once. We can continue this argument.

2. Now we consider the problem (1.1).
We shall say that $f(x) \equiv 0$ is the trivial solution of (1.1). If $s = 1$, then (1.1) holds for every polynomial. So we assume $s > 1$.

NOTATIONS. Let $C^{(s)}$ be the set of non-trivial solutions of (1.1) over the complex field, $C_n^{(s)}$ donote the solutions of degree n. Let $E(n)$ be the set of complex n'th unit roots, $\varepsilon_r = e^{ir2\pi/n}$.
It is obious that:

(A_1): If $f \in C^{(s)}$ and $g \in C^{(s)}$, then $f \cdot g \in C^{(s)}$.
(A_2): $C_0^{(s)} = E(s-1)$.

Let $f \in C_n^{(s)}$ $(n > 0)$,

$$f(x) = a_n \prod_{j=1}^{n} (x - \alpha_j) \quad (a_n \neq 0,\ a_n,\ \alpha_j \in C). \tag{2.1}$$

After the substitution $x \to \varepsilon_k x$, and by observing that

$$\prod_{k=1}^{s} (x - \bar{\varepsilon}_k \alpha_j) = x^s - \alpha_j^s,$$

from (1.1) we get

$$a_n^s \cdot \varepsilon_1^{ns(s+1)/2} \prod_{j=1}^{n} (x^s - \alpha_j^s) = a_n \prod_{j=1}^{n} (x^s - \alpha_j). \tag{2.2}$$

Hence we get that

(A$_3$): $a_n = \varepsilon_r (-1)^{n(s+1)}$ $[\varepsilon_r \in E(s-1) = C_0^{(s)}]$.

Let

$$D_n^{(s)} = \{f / f \in C_n^{(s)},\ a_n = (-1)^{n(s+1)}\},$$

$$D^{(s)} = \bigcup_{n \geq 0} D_n^{(s)}.$$

We shall say that the elements of $D^{(s)}$ are canonical solutions of (1.1). Furthermore each solution of degree n, can be written uniquely as the product of an element of $D_n^{(s)}$ and of $C_0^{(s)}$. Now we are interested only in canonical solutions.

For every polynomial f let $G(f)$ be the system of its roots: $G(f) = \{\alpha_1, \ldots, \alpha_n\}$.

The following assertion is obvious.

(A$_4$): If $f \in D_n^{(s)}$ $(n > 0)$, then $G(f)$ satisfies (1.3).

If $\{\alpha_1, \ldots, \alpha_n\}$ satisfies (1.3), then

$$f(x) = (-1)^{n(s+1)} \prod_{j=1}^{n} (x - \alpha_j)$$

is an element of $D_n^{(s)}$.

Definition 4. The polynomial $f \in D_n^{(s)}$ $(n > 0)$ is called elementary if $G(f)$ is a connected system.

Let $D_n^{(s)*}$ denote the set of elementary solutions.

Theorem 1. *If $f \in D_n^{(s)}$, then f can be written uniquely as a product of elementary solutions.*

Proof. It is a straightforward consequence of (A$_1$) and Lemma 4 and 5. ∎

Now we shall deal with elementary solutions. A system H containing only one element is connected only if $H = \{0\}, \{\varepsilon_1\}, \ldots, \{\varepsilon_{s-1}\}$, therefore

(A$_5$): $D_1^{(s)} = D_1^{(s)*} = \{(-1)^{s+1} x;\ (-1)^{s+1} (x - \varepsilon_j)\ (j = 1, \ldots, s-1)\}$.

Let $f \in D_n^{(s)*}$, $n > 1$. $f(\alpha) = 0$ implies that $\alpha^{s^n} - \alpha = 0$. Since $\alpha = 0$ belongs to 1, we have $\alpha^{s^n - 1} - 1 = 0$, i.e. $\alpha \in E(s^n - 1)$. Let $\alpha = \varepsilon_r \in E(s^n - 1)$. Then, from Lemma 4

$$G(f) = \{\varepsilon_r, \varepsilon_r^s, \ldots, \varepsilon_r^{s^{n-1}}\} = \{\varepsilon_r, \varepsilon_{rs}, \ldots, \varepsilon_{rs^{n-1}}\}.$$

DEFINITION 5. Let $r_1, \ldots, r_n$ be integers. The set $I(n) = \{r_1, \ldots, r_n\}$ is called a suitable index-set, if:

(a) $r_i \not\equiv r_j \pmod{(s^n - 1)}$ for $i \neq j$,

(b) with a suitable rearrangement

$$r_{i+1} \equiv r_i s \pmod{(s^n - 1)}.$$

Two suitable index-sets are identical if their elements after a suitable rearrangement are pairwise congruent mod $(s^n - 1)$.

The following assertion can be proved easily:

(A_6): $\{\varepsilon_{r_1}, \varepsilon_{r_2}, \ldots, \varepsilon_{r_n}\}$ $(\varepsilon_{r_i} \in E(s^n - 1))$ is connected if and only if $\{r_1, r_2, \ldots, r_n\}$ is a suitable index-set.

(A_7): $I_0(n) = \{1, s, s^2, \ldots, s^{n-1}\}$ is a suitable index-set.

(A_8): If $I(n)$ is a suitable index-set and $r \in I(n)$, then $I(n)$ is identical with $r I_0(n)$.

From (A_8) it follows that if the suitable index-sets I_1, I_2 have a common element mod $(s^n - 1)$, then they are identical.

LEMMA 6. $I(n) = r \cdot I_0(n)$ is a suitable index-set if and only if

$$r \not\equiv 0 \quad \left(\operatorname{mod} \left(\frac{s^n - 1}{s^d - 1}\right)\right)$$

if $d < n$, and d divides n.

PROOF. We have $r_i = r \cdot s^{i-1}$ $(i = 1, \ldots, n)$. Therefore (b) in Definition 5 is valid. The condition (a) holds if and only if

$$r \cdot (s^k - s^m) \not\equiv 0 \qquad (\operatorname{mod} (s^n - 1)) \tag{2.2}$$

is satisfied for every $(0 \leqq)\ m < k\ (< n)$. (2.2) is equivalent to

$$r \cdot (s^{k-m} - 1) \not\equiv 0 \qquad (\operatorname{mod} (s^n - 1)).$$

Observing that $(s^a - 1, s^b - 1) = s^{(a,b)} - 1$, we get the desired result immediately.

NOTATION. Let $H_n^{(s)}$ be the number of those integers r in $\{0, 1, \ldots, s^n - 1\}$ for which the relation

$$r \equiv 0 \quad \left(\operatorname{mod} \left(\frac{s^n - 1}{s^d - 1}\right)\right)$$

does not hold for $d | n$, $d < n$.

LEMMA 7. We have

$$|H_n^{(s)}| = \sum_{d|n} \mu\left(\frac{n}{d}\right) s^d. \tag{2.3}$$

PROOF. It is enough to see the equation

$$x^{s^n}-x=0$$

and classify its roots α, and use the inversion formula. ▌

THEOREM 2. *We have*

$$|D_n^{(s)*}| = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)s^d .$$

PROOF. We consider (2.3) and take into account that $r, r\cdot s, r\cdot s^2, \ldots, \ldots, r\cdot s^{n-1}$ generate the same index-set. ▌

3. Now we shall characterize the real solutions of (1.1).

NOTATION. Let $R^{(s)}$ denote the set of real solutions of (1.1), $R_n^{(s)}$ the subset of n'th degree polynomials, $R_n^{(s)*}\subseteq R_n^{(s)}$ the set of elementary polynomials satisfying $a_n=(-1)^{n(s+1)}$.

For a polynomial $f=\sum b_j x^j$ the conjugate $\bar{f}$ is defined by $\bar{f}=\sum \bar{b}_j x^j$.

DEFINITION 6. For a system $H=\{\alpha_1, \ldots, \alpha_n\}$, its conjugate is $\bar{H}= =\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$. We shall say that H is autoconjugate, if $H\doteq\bar{H}$.

The following statements are obvious.

(A_9): Let f be a polynomial with real a_n, and $\deg f\geqq 1$. The coeffitients of f are real if and only if $G(f)$ is autoconjugate.

(A_{10}): If f is elementary, then $\bar{f}$ is elementary too.

Let $f\in D_n^{(s)}$, and

$$f=\prod_{i=1}^{K} f_i \tag{3.1}$$

its factorization into elementary solutions. We shall prove that for $f\in R_n^{(s)}$ each factor f_i is contained in (3.1) with the same multiplicity as its conjugate.

Indeed, if $f\in R_n^{(s)}$, then $G(f)$ is autoconjugate. Let $\alpha\in G(f)$. Assume that α occurs in $G(f)$ k times. Since $G(f)$ is autoconjugate, therefore $\bar{\alpha}$ occurs in $G(f)$ k times. Let $f_1=\ldots=f_k$ be the elementary solutions generated by $\alpha: f_1(\alpha)=0$. Then there exist k elementary solutions in (3.1) generated by $\bar{\alpha}$.

The converse assertion is obvious. If $f\in D_n^{(s)}$ has a representation with this property, then $f\in R_n^{(s)}$.

So we have proved

THEOREM 3. $f\in R_n^{(s)}$ *if and only if each factor* f_i *is contained in* (3.1) *with the same multiplicity as its conjugate,* $\bar{f}_i$.

The following assertions are valid.

(A_{11}): $R_0^{(s)}=\begin{cases}\{-1, 1\}, & \text{if } s \text{ is odd}\\ \{1\}, & \text{if } s \text{ is even}\end{cases}.$

(A_{12}): $R_1^{(s)*} = \begin{cases} \{-x, -(x-1)\} & \text{if } s \text{ is even} \\ \{x, x+1, x-1\} & \text{if } s \text{ is odd} \end{cases}.$

(A_{13}): The system $\{\varepsilon_{r_1}, \varepsilon_{r_2}, \ldots, \varepsilon_{r_n}\}$ $(\varepsilon_{r_i} \in E(s^n-1))$ is a connected autoconjugate system if and only if for the corresponding index-set $I(n) = \{r_1, r_2, \ldots, r_n\}$ the relation $I(n) = -I(n)$ is satisfied.

NOTATION. Let $K_{2m}^{(s)}$ be the set of those elements in $H_{2m}^{(s)}$ which are divisible by (s^m-1).

LEMMA 8. Let $n>1$, $I(n) = r \cdot I_0(n)$ be an index-set with n elements. The relation $I(n) = -I(n)$ holds if and only if $n = 2m$ and $r \in K_{2m}^{(s)}$.

PROOF. 1. Assume that $n>1$ being odd and $I(n) = -I(n)$. Then for a suitable k $r \cdot s^k \equiv -r \cdot s^k \pmod{s^n-1}$ whence $2r \equiv 0 \pmod{s^n-1}$, i.e.

$$r \equiv 0 \begin{cases} \bmod \left(\dfrac{s^n-1}{s-1}(s-1)\right) & \text{if } s \text{ is even} \\ \bmod \left(\dfrac{s^n-1}{s-1}\,\dfrac{s-1}{2}\right) & \text{if } s \text{ is odd} \end{cases}$$

that contradicts to Lemma 6. So n has to be even.

2. Let $n = 2m$, and ν be defined by

$$r \equiv -r \cdot s^\nu \pmod{s^{2m}-1} \quad (1 \leqq \nu < 2m). \tag{3.2}$$

Hence

$$r(s^{2\nu}-1) \equiv 0 \pmod{s^{2m}-1},$$

and so

$$r \equiv 0 \left(\bmod \frac{s^{2m}-1}{s^d-1}\right),$$

where $d = (2m, 2\nu)$. By using Lemma 6 we get that $d = 2m$, i.e. $\nu = m$. Consequently $r \equiv 0 \pmod{s^m-1}$ is a necessary condition.

The sufficiency is obvious. ∎

As a straightforward corollary we have

THEOREM 4.

$$|R_{2m}^{(s)*}| = \frac{|K_{2m}^{(s)}|}{2m}.$$

4. Let us consider finally the rational solutions.

Let $f(x)$ be a rational solution, $f(\alpha) = 0$. Since α is a complex unit-root, therefore $f(x)$ is a multiple of the cyclotomic polynomial generated by α. First we deal with irreducible solutions of (1.1).

THEOREM 5. Let $n>1$, and f be rational irreducible polynomial of degree n, $a_n = (-1)^{n(s+1)}$. f is a solution of (1.1) if and only if there exist positive integers h and k such that $h|(s^k-1)$ and $h \in H_k^{(s)}$ for $k>1$, furthermore if f is the cyclotomic polynimial of rank $(s^k-1)/h$.

PROOF. 1. Assume that f is a rational irreducible solution of (1.1), with $a_n = (-1)^{n(s,1)}$. Assume that in its factorization (according to Theorem 1) there is an elementary factor of degree k. Then there exists $\varepsilon_r \in E(s^k-1)$ such that $f(\varepsilon_r)=0$. Let $h=(r, s^k-1)$, i.e. $(r/h, (s^k-1)/h)=1$. This means that ε_r is a primitive root of rank $(s^k-1)/h$. Since n is even therefore $a_n=1$.

Since ε_r is a root of the elementary solution of degree k, for $k>1$ we have $r\in H_k^{(s)}$ (see Lemma 6), and for $h|r$ we get $h\in H_k^{(s)}$. Consequently the conditions are necessary.

2. Now we prove that the conditions are sufficient. Since $n = \varphi((s^k-1)/h)>1$, therefore $(s^k-1)/h>2$ holds.

(a) (Case $k=1$). Let $h|s-1$, $(s-1)/h>2$ and f be the cyclotomic polynomial of rank $(s-1)/h$. Since the roots of f are elements of $E(s-1)$, therefore f is a product of $\varphi((s-1)/h)$ (even!) linear polynomials. From (A_1) we have that f is a solution.

(b) (Case $k>1$). Let $h|(s^k-1)$, $(s^k-1)/h>2$, $h\in H_k^{(s)}$ and f be the cyclotomic polynomial of rank $(s^k-1)/h$. The roots of f are the primitive roots of degree $(s^k-1)/h$, i.e. if $f(\varepsilon_r)=0$ then $\varepsilon_r\in E((s^k-1)/h)$ and $(r, (s^k-1)/h)=1$. Let J denote the index-set derived from the roots of f. J is a reduced residue system mod (s^k-1). We shall prove that $h\cdot J\subseteq H_k^{(s)}$. It is enough to show that from $h\in H_k^{(s)}$, $h|(s^k-1)$, $(u, (s^k-1)/h)=1$ it follows that the relation

$$h\cdot u\equiv 0 \quad \left(\operatorname{mod}\left(\frac{s^k-1}{s^d-1}\right)\right) \tag{4.1}$$

has no solution if d is a divisor of k, $d<k$. Assume that (4.1) has a solution. Let $h_1=(h, (s^k-1)/(s^d-1))$ and $h=h_1\cdot h_2$. Since $h|(s^k-1)$, therefore $h_2|(s^d-1)$. From (4.1) we have

$$u\equiv 0 \quad \left(\operatorname{mod}\left(\frac{s^k-1}{h_1(s^d-1)}\right)\right). \tag{4.2}$$

Since

$$\frac{s^k-1}{h_1(s^d-1)} = \frac{s^k-1}{h\,\dfrac{s^d-1}{h_2}} \;\Big|\; \frac{s^k-1}{h},$$

therefore from (4.2) $\dfrac{s^k-1}{h_1(s^d-1)}=1$, whence $\dfrac{s^k-1}{s^d-1}h_2=h$, that contradicts to the definition of h.

It is easy to see furthermore that from $r\in h\cdot J$ we have $r\cdot I_0(k)\subseteq h\cdot J$. Taking into account ($A_7$), ($A_8$) and Lemma 6, $h\cdot J$ can be stated as the union of index-sets $h\cdot u\cdot I_0(k)$, i.e. J is a union of index-sets $u\cdot I_0(k)$.

Since the degree of f is even, f is a product of elementary solution, i.e. $f\in D_n^{(s)}$. f is irreducible. The sufficiency has been proved. ∎

We remark that as a by-product we have got the following results:

1. If $h|(s^k-1)$ and $h\in H_k^{(s)}$, then $k|\varphi((s^k-1)/h)$.

2. $\sum_{h_i} \varphi\left(\frac{s^k-1}{h_i}\right) = \sum_{d|n} \mu\left(\frac{k}{d}\right) s^d$

where the summation is extended for every divisor of (s^k-1) belonging to $H_k^{(s)}$.

(For a special case of 1. see [4]).

THEOREM 6. *The rational polynomial f of degree n with* $a_n=(-1)^{n(s+1)}$ *is a solution of* (1.1) *if and only if it can be written in the form*

$$f=[(-1)^{s+1}x]^{n_1}[(-1)^{s+1}(x+1)]^{n_2}[(-1)^{s+1}(x-1)]^{n_3}\prod_j h_j \tag{4.3}$$

where the factors h_j *are cyclotomic polynomials satisfying the conditions stated for f in Theorem 5.*

PROOF. We need to prove only that there no other solution exists. Let f be a rational solution of degree n, and $f=\prod_i h_i$ its factorization into irreducible factors. $h_i(\alpha)=0$ involves that $f(\alpha)=0$. If α is real then $\alpha\in\in\{-1, 0, 1\}$ and so is h_i of degree 1. If the degree of it is greater than 1, then it belongs to a k. Arguing as in the proof of Theorem 5, we get that h_i is a cyclotomic polynomial having the properties stated in Theorem 5. Consequently

$$f=(-1)^{n(s+1)}x^{n_1}(x+1)^{n_2}(x-1)^{n_3}\prod_j h_j$$

where h_j are suitable cyclotomic polynomials each having an even degree. The parity of n is the same as of $n_1+n_2+n_3$. The proof has been completed. ∎

We have proved that each solution has integer coefficients, since the coefficients of the cyclotomic polynomials are integers.

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NOTE ON SUPER PSEUDOPRIME NUMBERS

By

J. FEHÉR and P. KISS

Teacher's Training College, Pécs and Teacher's Training College, Eger

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Let n be a positive composite integer. If

$$n|(a^n - a)$$

for an integer $a > 1$, then n is called pseudoprime with respect to a. It is called super pseudoprime with respect to a if each divisor of it is a prime or pseudoprime with respect to a. In the case $a = 2$ the namings are only pseudoprime and super pseudoprime.

The pseudoprime numbers were studied by several authors and the results up to 1971 were collected by E. Lieuwens [3] and A. Rotkiewicz [4]. For example, we know that there are infinitely many super pseudoprimes. K. Szymiczek [6] showed that $F_n F_{n+1}$ is super pseudoprime for $n > 1$, where $F_n = 2^{2^n} + 1$ is the n^{th} Fermat number. A. Rotkiewicz [5] proved that there are infinitely many super pseudoprimes which are products of exactly three distinct primes.

The purpose of this note is to show that the Rotkiewicz's result mentioned above is valid for super pseudoprimes with respect to certain integer a, too. We prove the following theorem.

Theorem. *Let a be an integer with conditions $a > 1$ and $4 \nmid a$. Then there exist infinitely many super pseudoprime numbers with respect to a which are products of exactly three distinct primes.*

For the proof we need two lemmas.

Lemma 1. There are infinitely many primes p of the form $4k-1$ for which $2p+1$ is a composite number.

Lemma 2. Let a be an integer with conditions $a > 1$ and $4 \nmid a$. Then there are infinitely many primes p of the form $4k-1$ for which $p|(a^d - 1)$, where d is an integer and $d < \frac{p-1}{2}$.

PROOF OF LEMMA 1. Let us suppose that $p_1, \ldots, p_r$ are all the primes satisfying the conditions of Lemma 1. Let p be a prime of the form $4k-1$ greater than the p_i's. Consider the sequence $\{q_n\}$ defined by $q_0 = p$ and $q_n = 2q_{n-1}+1$ for $n>0$. Each term of this sequence has the form $4k-1$ and each term is a prime by our supposition. But it is easy to show by induction on n that $q_n = 2^n p+2^n-1$ and therefore q_{p-1} cannot be a prime since $q_{p-1} \equiv 0 \pmod p$ by Fermat's congruence theorem. This contradiction proves the lemma. (We note that this result as a problem was proposed by K. GYŐRY [2]).

PROOF OF LEMMA 2. Let d be a prime of the form $4k-1$ for which $2d+1$ is a composite number. By Lemma 1, infinitely many primes exist with such properties. Let us suppose that $a \equiv 1$ or $2 \pmod 4$ and $(d, a-1) = 1$. In this case

$$\left(\frac{a^d-1}{a-1}, a-1\right) = 1 .$$

Indeed, $q \left| \frac{a^d-1}{a-1} \right.$ and $q|(a-1)$ would imply the congruence

$$0 \equiv \frac{a^d-1}{a-1} = a^{d-1}+a^{d-2}+\ldots+a+1 \equiv d \pmod q$$

which contradicts to the condition $(d, a-1) = 1$. It is easy to see that $(a^d-1)/(a-1)$ is a number of the form $4k-1$ and each prime factor of it has the form $kd+1$. From these it follows that there is a prime p of the form $4k-1$ for which $p|(a^d-1)$, $(p, a-1) = 1$ and $p = k_1 d+1$, where k_1 is an integer. But $k_1 > 2$ since, by the conditions, in cases $k_1 = 1$ and $k_1 = 2$ the number $k_1 d+1$ is not a prime. Thus $d < \frac{p-1}{2}$. Distinct d's determine distinct p's since $(a^m-1, a^n-1) = a^{(m,n)}-1$. Therefore the lemma is true in cases $a = 1$ and $2 \pmod 4$.

We can similarly prove the lemma in case $a \equiv -1 \pmod 4$. Let now d be a prime of the form $4k-1$ for which $(d, a^2-1) = 1$ and $2d+1$ is a composite number. In this case $(a^{2d}-1)/(a^2-1)$ has the form $4k-1$ and has a prime divisor p of the form $4k-1$ for which $(p, a^2-1) = 1$, furthermore $p = k_2 d+1$ for some integer k_2. Here $k_2 > 4$ since otherwise p would be composite or would have the form $4k+1$, thus $2d < \frac{p-1}{2}$. From this, similarly as above, the statement follows.

PROOF OF THE THEOREM. Let $p\,(>12)$ be a prime of the form $4k-1$ and let $a\,(>1)$ be an integer with condition $4 \nmid a$. Let us suppose that a belongs to the exponent d modulo p, where $d < \frac{p-1}{2}$. Let q and r be primes which are primitive prime divisors of the numbers $a^{\frac{p-1}{2}}-1$ and $a^{p-1}-1$ respectively.

Such primes q and r exist by a theorem of K. ZSIGMONDI [7] or of G. D. BIRKHOFF and H. S. VANDIVER [1] if $a \neq 2$ or $\frac{p-1}{2} > 6$. We show that $n_0 = pqr$ is super pseudoprime with respect to a.

By the choice of q and r, we have $q = k_3 \cdot \frac{p-1}{2} + 1$ and $r = k_4(p-1)+1$ for some integers k_3 and k_4. But k_3 cannot be an odd number since otherwise q would not be prime by the form of p, therefore $q = k_5(p-1)+1$, where k_5 is an integer. From this it follows that each of the numbers n_0, $n_1 = pq$, $n_2 = pr$ and $n_3 = qr$ has the form $k(p-1)+1$ and so each of the primes p, q, r is a divisor of number $a^{n_i-1}-1$ for $i = 0, 1, 2$ and 3. It implies that n_0, n_1, n_2 and n_3 are pseudoprimes with respect to a. Thus n_0 is super pseudoprime with respect to a which, together with Lemma 1 and 2, proves the theorem.

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LINEAR FUNCTIONALS ON HARDY SPACES

By

J. MOGYORÓDI

Department of Probability Theory, L. Eötvös University, Budapest

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1. Let Φ be a Young-function. For the definition and the properties of this we refer to [1] and [2].

DEFINITION 1. We say that the random variable X defined on the probability space $(\Omega, \mathcal{A}, P)$ belongs to the so called Orlicz space $L^\Phi(\Omega, \mathcal{A}, P)$ if there is a constant $a>0$ such that

$$E\left(\Phi\left(\frac{|X|}{a}\right)\right) \leqq 1 .$$

The L^Φ-norm of $X \in L^\Phi$ is defined as

$$\|X\|_\Phi = \inf\left(a : a>0, \quad E\left(\Phi\left(\frac{|X|}{a}\right)\right) \leq 1\right).$$

The normed vector space L^Φ is complete. ([2]).

Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ be a sequence of σ-fields of events such that

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{n=0}^{\infty} \mathcal{F}_n\right) = \mathcal{A} .$$

Consider the random variable $X \in L_1$ and the martingale

$$X_n = E(X \mid \mathcal{F}_n), \quad n \geqq 0 ,$$

where we suppose that $X_0 = 0$ a.e. Denote by $\{d_i\}$ the corresponding martingale-differences.

DEFINITION 2. We say that the random variable $X \in L_1$ belongs to the Hardy space $\mathcal{H}_\Phi$ generated by the Young-function Φ if the random variable

$$S = \left(\sum_{i=1}^{\infty} d_i\right)^{1/2}$$

belongs to L^Φ.

We define

$$\|X\|_{\mathcal{H}_\Phi} = \|S\|_\Phi .$$

The aim of the present note is to characterize the bounded linear functionals defined on the Hardy space $\mathcal{H}_\Phi$.

2. This section contains some introductory remarks. The following assertion is known.

LEMMA 1. Let (Φ, Ψ) be a pair of conjugate Young-functions. Let us suppose that the quantity

$$q = \sup_{x>0} \frac{x\,\psi(x)}{\Psi(x)},$$

called the power of the Young-function $\Psi(x)$ [where $\psi(x)$ denotes the right-hand side derivative of Ψ], is finite. Then any linear and bounded functional $L(Y)$, defined for arbitrary $Y \in L^\Psi$, is of the form

$$L(Y) = E(XY),$$

where $X \in L^\Phi$ and we have

$$\|X\|_\Phi \le \|L\| .$$

The following definition and lemma are also needed to get our purpose.

DEFINITION 3. Let Ψ be a Young-function. We say that the sequence

$$\Theta = (\Theta_1, \Theta_2, \ldots)$$

of random variables belongs to the Banach space $\delta\mathcal{H}_\Psi$ if

$$\left(\sum_{i=1}^{\infty} \Theta_i^2\right)^{1/2} \in L^\Psi .$$

In this case we define

$$\|\Theta\|_{\delta\mathcal{H}_\Psi} = \left\|\left(\sum_{i=1}^{\infty} \Theta_i^2\right)^{1/2}\right\|_\Psi .$$

LEMMA 2. Let $\lambda(\Theta)$ be a linear functional on the elements of $\delta\mathcal{H}_\Psi$ and suppose that

$$|\lambda(\Theta)| \le B\,\|\Theta\|_{\delta\mathcal{H}_\Psi},$$

where $B > 0$ is a constant. Suppose also that the power q is finite. Then in the Banach space $\delta\mathcal{H}_\Phi$ there exists an element

$$\sigma = (\sigma_1, \sigma_2, \ldots) \in \delta\mathcal{H}_\Phi$$

such that

$$\lambda(\Theta) = \lim_{n\to+\infty} \sum_{i=1}^{n} E(\sigma_i \Theta_i)$$

and

$$\|\sigma\|_{\delta\mathcal{H}_\Phi} E\left(\Phi\left(\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2} \Big/ \|\sigma\|_{\delta\mathcal{H}_\Phi}\right)\right) \le B . \tag{1}$$

Here Φ is the conjugate of Ψ.

PROOF. Let $X\in L^{\Psi}$. For arbitrary $n\geqq 1$ we put

$$\Theta_i=\begin{cases}X, & \text{if} \quad i=n\\ 0, & \text{if} \quad i\neq n\end{cases}$$

and we take

$$\lambda_n(X)=\lambda(\Theta).$$

Trivially, $\Theta\in\delta\mathscr{H}_{\Psi}$ and so, by our supposition,

$$|\lambda_n(X)|=|\lambda(\Theta)|\leq B\,\|\Theta\|_{\delta\mathscr{H}_{\Psi}}=B\,\|X\|_{\Psi}\,.$$

The power q of Ψ being finite by the preceding lemma there exists $\sigma_n\in L^{\Phi}$ such that

$$\lambda(\Theta)=\lambda_n(X)=E(\sigma_n X)\,.$$

Consider now a sequence

$$\Theta^{(n)}=(\Theta_1,\,\ldots,\,\Theta_n,\,\ldots)\in\delta\mathscr{H}_{\Psi}\,,$$

for which $\Theta_i=0$, if $i>n$. By the linearity of the functional λ we have

$$\lambda(\Theta^{(n)})=\sum_{i=1}^{n}E(\sigma_i\,\Theta_i)\,,$$

where $\sigma_i\in L^{\Phi}$, $i=1,2,\ldots,n$. Since

$$\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}\leqq\sum_{i=1}^{n}|\sigma_i|$$

it follows that

$$\left\|\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}\right\|_{\Phi}\leqq\sum_{i=1}^{n}\|\sigma_i\|_{\Phi}<+\infty$$

and consequently,

$$\sigma^{(n)}=(\sigma_1,\sigma_2,\,\ldots,\,\sigma_n,0,0,\,\ldots)\in\delta\mathscr{H}_{\Phi}\,.$$

We can suppose that $\|\sigma^{(n)}\|_{\delta\mathscr{H}_{\Phi}}$ does not vanish if n is large.

Now we prove the validity of (1) for $\sigma^{(n)}$, where n is arbitrary. Let us take Θ_i in the form $a_n\sigma_i$, where the random variable $a_n\geqq 0$ is determined by the postulate that

$$\Theta^{(n)}=(a_n\sigma_1,\,\ldots,\,a_n\sigma_n,0,0,\,\ldots)$$

belong to $\delta\mathscr{H}_{\Psi}$ and that the value

$$\lambda(\Theta^{(n)})=\sum_{i=1}^{n}E(\sigma_i\,\Theta_i)=E\left\{\left(\sum_{i=1}^{n}\sigma_i^2\right)a_n\right\}$$

of the functional λ be equal to the expression

$$\|\sigma^{(n)}\|_{\delta\mathscr{H}_{\Phi}}\,E\left\{\Phi\left(\frac{\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}}{\|\sigma^{(n)}\|_{\delta\mathscr{H}_{\Phi}}}\right)\right\}.$$

On the event $A=\left\{\sum_{i=1}^{n}\sigma_i^2>0\right\}$ let us determine a_n by the formula

$$a_n=\|\sigma^{(n)}\|_{\delta\mathcal{H}_\Phi}\,\Phi\left(\frac{\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}}{\|\sigma^{(n)}\|_{\delta\mathcal{H}_\Phi}}\right)\Bigg/\sum_{i=1}^{n}\sigma_i^2\,,$$

while on the complementary set take $a_n=0$. Then on the event A we have

$$\left(\sum_{i=1}^{n}\Theta_i^2\right)^{1/2}=a_n\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}=$$

$$=\|\sigma^{(n)}\|_{\delta\mathcal{H}_\Phi}\,\Phi\left(\frac{\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}}{\|\sigma^{(n)}\|_{\delta\mathcal{H}_\Phi}}\right)\Bigg/\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}$$

and on the complementary one

$$\left(\sum_{i=1}^{n}\Theta_i^2\right)^{1/2}=0\,.$$

Since for $x>0$ we have

$$\Psi\left(\frac{\Phi(x)}{x}\right)\leqq\Phi(x)\,,$$

we obtain that

$$E\left(\Psi\left(\left(\sum_{i=1}^{n}\Theta_i^2\right)^{1/2}\right)\right)\leqq E\left\{\Psi\left\{\frac{\Phi\left(\dfrac{\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}}{\|\sigma^{(n)}\|_{\delta\mathcal{H}_\Phi}}\right)}{\dfrac{\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}}{\|\sigma^{(n)}\|_{\delta\mathcal{H}_\Phi}}}\right\}\right\}\leqq$$

$$\leqq E\left\{\Phi\left(\frac{\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}}{\|\sigma^{(n)}\|_{\delta\mathcal{H}_\Phi}}\right)\right\}\leqq 1\,.$$

Consequently,

$$\left(\sum_{i=1}^{n}\Theta_i^2\right)^{1/2}\in L^\Psi\,,$$

since

$$E\left(\Psi\left(\left(\sum_{i=1}^{n}\Theta_i^2\right)^{1/2}\right)\right)\leqq 1\,.$$

It follows that

$$\Theta^{(n)}=(a_n\sigma_1,\,a_n\sigma_2,\,\ldots,\,a_n\sigma_n,\,0,\,0,\,\ldots)\in\delta\mathcal{H}_\Psi$$

and that

$$\|\Theta^{(n)}\|_{\delta \mathcal{H}_\Psi} \leqq 1 .$$

On the other hand by the boundedness of λ we have

$$\lambda(\Theta^{(n)}) = \|\sigma^{(n)}\|_{\delta \mathcal{H}_\Phi} E\left(\Phi\left(\left(\sum_{i=1}^{n} \sigma_i^2\right)^{1/2} \Big/ \|\sigma^{(n)}\|_{\delta \mathcal{H}_\Phi}\right)\right) \leqq B \|\Theta^{(n)}\|_{\delta \mathcal{H}_\Psi} .$$

So, by remarking that $\|\Theta^{(n)}\|_{\delta \mathcal{H}_\Psi} \leqq 1$ we get

$$\|\sigma^{(n)}\|_{\delta \mathcal{H}_\Phi} E\left(\Phi\left(\left(\sum_{i=1}^{n} \sigma_i^2\right)^{1/2} \Big/ \|\sigma^{(n)}\|_{\delta \mathcal{H}_\Phi}\right)\right) \leqq B$$

for arbitrary finite n.

Let now $\gamma \in \delta \mathcal{H}_\Psi$ be arbitrary. Then, if $n \to +\infty$,

$$\left\|\left(\sum_{i=n+1}^{\infty} \gamma_i^2\right)^{1/2}\right\|_\Psi \to 0 .$$

From this by the boundedness and linearity of the functional λ with

$$\gamma^{(n)} = (\gamma_1, \gamma_2, \ldots, \gamma_n, 0, 0, \ldots)$$

we get for $n \to +\infty$

$$|\lambda(\gamma) - \lambda(\gamma^{(n)})| \leqq B \left\|\left(\sum_{i=n+1}^{\infty} \gamma_i^2\right)^{1/2}\right\|_\Psi \to 0 .$$

Consequently,

$$\lambda(\gamma) = \lim_{n\to+\infty} \lambda(\gamma^{(n)}) = \lim_{n\to+\infty} \sum_{i=1}^{n} E(\sigma_i \gamma_i) ,$$

where, as we have seen above,

$$\|\sigma^{(n)}\|_{\delta \mathcal{H}_\Phi} E\left[\Phi\left(\frac{\left(\sum_{i=1}^{n} \sigma_i^2\right)^{1/2}}{\|\sigma^{(n)}\|_{\delta \mathcal{H}_\Phi}}\right)\right] \leqq B, \quad n = 1, 2, \ldots .$$

From this

$$\|\sigma\|_{\delta \mathcal{H}_\Phi} E\left(\Phi\left(\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2} \Big/ \|\sigma\|_{\delta \mathcal{H}_\Phi}\right)\right) \leqq B$$

and this means that

$$\sigma = (\sigma_1, \sigma_2, \ldots) \in \delta \mathcal{H}_\Phi$$

with

$$\|\sigma\|_{\delta \mathcal{H}_\Phi} E\left(\Phi\left(\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2} \Big/ \|\sigma\|_{\delta \mathcal{H}_\Phi}\right)\right) \leqq B .$$

This proves the assertion.

It should be mentioned that when Φ has finite power than for arbitrary $X \in L_\Phi$ such that $\|X\|_\Phi > 0$ we have

$$E\left(\Phi\left(\frac{\|X\|}{\|X\|_\Phi}\right)\right) = 1 .$$

From this the following consequence of the preceding lemma can be deduced

COROLLARY 1. Let $\lambda(\Theta)$ be a linear functional on the elements Θ of $\delta\mathcal{H}_\Psi$ and suppose that

$$|\lambda(\Theta)| \leqq B \|\Theta\|_{\delta\mathcal{H}_\Psi} ,$$

where $B > 0$ is a constant. Suppose that Φ and Ψ have finite power. Then in the Banach space $\delta\mathcal{H}_\Phi$ there exists an element

$$\sigma = (\sigma_1, \sigma_2, \ldots) \in \delta\mathcal{H}_\Phi$$

such that

$$\lambda(\Theta) = \lim_{n \to +\infty} \sum_{i=1}^{n} E(\sigma_i \Theta_i)$$

and

$$\|\sigma\|_{\delta\mathcal{H}_\Phi} \leqq B .$$

3. Consider the pair (Φ, Ψ) of conjugate Young-functions, or let $\Phi(x) = cx$, where $c > 0$ is a constant. Let

$$\Phi'(x) = \Phi(x^2) .$$

Then as it is easily seen, Φ' is also a Young-function. We shall denote by Ψ' its conjugate Young-function.

DEFINITION 4. Let $X \in L_2$ and consider the corresponding martingale $(X_n, \mathcal{F}_n)$, $n \geqq 0$, introduced in section 1. We say that $X \in \mathcal{K}_{\Phi'}$, if there exists $\gamma \in L^{\Phi'}$ such that

$$E\left((Y - X_{n-1})^2 | \mathcal{F}_n\right) = \sum_{i=n}^{\infty} E(d_i^2 | \mathcal{F}_n) \leqq E(\gamma^2 | \mathcal{F}_n)$$

holds a.e. for all $n \geqq 1$.

Let $X \in \mathcal{K}_{\Phi'}$ and consider the following class of random variables

$$\Gamma_X^{\Phi'} = \{\gamma : \gamma \in L^{\Phi'}, \quad E\left((X - X_{n-1})^2 | \mathcal{F}_n\right) \leqq E(\gamma^2 | \mathcal{F}_n) \quad \text{a.e.,} \quad n \geqq 1\} .$$

If $\Gamma_X^{\Phi'}$ is not empty then define

$$\|X\|_{\mathcal{K}_{\Phi'}} = \inf_{\gamma \in \Gamma_X^{\Phi'}} \|\gamma\|_{\Phi'} .$$

It can be easily seen that $\|X\|_{\mathcal{K}_{\Phi'}}$ is a quasi-norm.

The interesting case in this definition is when Φ' has no finite power. In this case $L^{\Phi'}$ is "near" to L_∞ and so the space $\mathcal{K}_{\Phi'}$ is "near" to the well-known BMO-space.

A class of the elements of $\mathcal{K}_{\Phi'}$ is characterized by the following

THEOREM 1. *Let*

$$\sigma = (\sigma_1, \sigma_2, \ldots)$$

be a sequence of random variables such that $\sigma \in \delta\mathcal{H}_{\Phi'}$. *Suppose further that* Ψ', *the conjugate of* Φ', *has finite power* q. *Then the martingale* $(Y_n, \mathcal{F}_n)$, $n \geqq 0$, *defined by the formulas* $Y_0 = 0$ a.e. *and*

$$Y_n = \sum_{i=1}^{n} \left(E(\sigma_i | \mathcal{F}_i) - E(\sigma_i | \mathcal{F}_{i-1})\right), \quad n \geqq 1$$

converges a.e. *and in the* $\mathcal{K}_{\Phi'}$*-quasi-norm to a random variable* Y. *We have*

$$\|Y\|_{\mathcal{K}\Phi'} \leq 2q \|\sigma\|_{\delta\mathcal{H}_{\Phi'}}.$$

PROOF. Let $\|\sigma\|_{\delta\mathcal{H}_{\Phi'}} = B$. GARSIA [3] has shown that the inequality

$$\sum_{i=n}^{\infty} E(d_i^2 | \mathcal{F}_n) \leqq E(\gamma^2 | \mathcal{F}_n), \quad n \geqq 1,$$

holds a.e., where

$$d_i = E(\sigma_i | \mathcal{F}_i) - E(\sigma_i | \mathcal{F}_{i-1})$$

and

$$\gamma = 2g^* = 2 \sup_{n \geqq 0} E(g | \mathcal{F}_n), \quad g = \left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2}.$$

We have to prove that γ belongs to $L^{\Phi'}$. To this end remark that by the well-known Doob-Garsia inequality [4] and by that of Jensen

$$E\left(\Phi'\left(\frac{\gamma}{2q\,B}\right)\right) = E\left(\Phi'\left(\sup_{n \geqq 0} E\left(\frac{g}{q\,B} \Big| \mathcal{F}_n\right)\right)\right) \leqq$$

$$\leqq \sup_{n \geqq 0} E\left(\Phi'\left(E\left(\frac{g}{B} \Big| \mathcal{F}_n\right)\right)\right) \leqq E\left(\Phi'\left(\frac{g}{B}\right)\right) \leqq 1.$$

Consequently, $\gamma \in L^{\Phi'}$ and $\|\gamma\|_{\Phi'} \leqq 2q\,B$.

As it is well-known, there is a constant $c_1 > 0$ such that

$$c_1 E\left(\sum_{i=1}^{\infty} d_i^2\right) \leqq c_1 E(\gamma^2) \leqq \|\gamma^2\|_{\Phi} = \|\gamma\|_{\Phi'}^2 \leqq 4q^2 B^2,$$

(cf. e.g. Proposition A–2–2 of [2]), or, in other words, $(Y_n, \mathcal{F}_n)$ is an L_2-bounded martingale. Thus

$$Y = \lim_{n \to +\infty} Y_n = \sum_{i=1}^{\infty} \left(E(\sigma_i | \mathcal{F}_i) - E(\sigma_i | \mathcal{F}_{i-1})\right)$$

exists a.e. and also Y_n converges to Y in L_2. Since

$$E\left(\sum_{i=n}^{\infty} d_i^2 | \mathcal{F}_n\right) = E\left((Y - Y_{n-1})^2 | \mathcal{F}_n\right) \quad \text{a.e.}$$

and we have cited above that

$$E\left(\sum_{i=n}^{\infty} d_i^2 | \mathcal{F}_n\right) \leqq E(\gamma^2 | \mathcal{F}_n) \quad \text{a.e.}$$

with

$$\gamma = 2 \sup_{n \geqq 0} E(g | \mathcal{F}_n)$$

it follows that $Y \in \mathcal{K}_{\Phi'}$ and that

$$\|Y\|_{\mathcal{K}_{\Phi'}} \leq 2q\, B\,.$$

Finally, we prove that $Y_n \to Y$ in the $\mathcal{K}_{\Phi'}$-sense. In fact,

$$Y - Y_n = \sum_{i=n+1}^{\infty} d_i$$

and for arbitrary $k \geqq n+1$ we have

$$\sum_{i=k}^{\infty} E(d_i^2 | \mathcal{F}_h) \leqq E(\gamma_n^2 | \mathcal{F}_h)\,,$$

where

$$\gamma_n = 2 \sup_{h \geqq n+1} E(g_n | \mathcal{F}_h), \quad g_n = \left(\sum_{i=n+1}^{\infty} \sigma_i^2\right)^{1/2}.$$

This can be proved in the same way as in the case of $n = 0$. Also, it can be shown that

$$E\left(\Phi'\left(\frac{\gamma_n}{2q\, B_n}\right)\right) \leqq 1$$

with

$$B_n = \left\|\left(\sum_{i=n+1}^{\infty} \sigma_i^2\right)^{1/2}\right\|_{\Phi'}.$$

Consequently, $Y - Y_n \in \mathcal{K}_{\Phi'}$ and

$$\|Y - Y_n\|\, \mathcal{K}_{\Phi'} \leqq \|\gamma_n\|_{\Phi'} \leq 2q\, B_n\,.$$

Since $B_n \to 0$ as $n \to +\infty$ it follows that $Y_n \to Y$ in the $\mathcal{K}_{\Phi'}$-sense. This proves the assertion.

We are now in the position to prove the main result of the present paper.

THEOREM 2. *Let $L(X)$ be a linear functional on the elements X of the space $\mathcal{H}_{\Psi'}$ and suppose that*

$$|L(X)| \leqq B\, \|X\|_{\mathcal{H}_{\Psi'}}$$

holds with some constant $B > 0$. Suppose also that the power q of Ψ' is finite. Then there exists a random variable $Y \in \mathcal{K}_{\Phi'}$ such that

$$L(X) = \lim_{n \to +\infty} E(X_n Y_n)\,.$$

PROOF. $\mathcal{H}_{\Psi'}$ is a subset of $\delta\mathcal{H}_{\Psi'}$. Namely, to arbitrary $X \in \mathcal{H}_{\Psi'}$ we can order the sequence

$$\Theta = (\Theta_1, \Theta_2, \dots) \in \delta\mathcal{H}_{\Psi'}$$

such that

$$\Theta_i = d_i = E(X|\mathcal{F}_i) - E(X|\mathcal{F}_{i-1}), \quad i \geq 1 .$$

Thus $L(X)$ is a linear and bounded functional on this subset. The Hahn–Banach theorem implies that $L(X)$ can be extended to a linear functional $\lambda(\Theta)$ on $\delta\mathcal{H}_{\Psi'}$ having the same bound as $L(X)$. By Lemma 2 we see that there is a $\sigma \in \delta\mathcal{H}_{\Phi'}$ such that

$$\|\sigma\|_{\delta\mathcal{H}_{\Phi'}} E\left(\Phi\left(\left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{1/2} \Big/ \|\sigma\|_{\delta\mathcal{H}_{\Phi'}}\right)\right) \leq B$$

and

$$\lambda(\Theta) = \lim_{n\to+\infty} \sum_{i=1}^{n} E(\sigma_i \Theta_i) .$$

Let $X \in \mathcal{H}_{\Psi'}$ be arbitrary and consider $X_n = \sum_{i=1}^{n} d_i$ $n \geq 0$, where $X_0 = 0$ a.e. Then trivially $X_n \in \mathcal{H}_{\Psi'}$ and for this random variable we have

$$\lambda(\Theta) = L(X_n) = \sum_{i=1}^{n} E\Big(\sigma_i\left(E(X|\mathcal{F}_i) - E(X|\mathcal{F}_{i-1})\right)\Big) =$$

$$= \sum_{i=1}^{n} E\Big(X\left(E(\sigma_i|\mathcal{F}_i) - E(\sigma_i|\mathcal{F}_{i-1})\right)\Big) =$$

$$= E\left(X_n \sum_{i=1}^{n} \left(E(\sigma_i|\mathcal{F}_i) - E(\sigma_i|\mathcal{F}_{i-1})\right)\right).$$

The random variables $Y_0 = 0$ and

$$Y_n = \sum_{i=1}^{n} \left(E(\sigma_i|\mathcal{F}_i) - E(\sigma_i|\mathcal{F}_{i-1})\right), \quad n \geq 1 ,$$

form a martingale and by Theorem 1 it follows that

$$Y = \lim_{n\to+\infty} Y_n = \sum_{i=1}^{\infty} \left(E(\sigma_i|\mathcal{F}_i) - E(\sigma_i|\mathcal{F}_{i-1})\right) \in \mathcal{K}_{\Phi'}$$

and we have

$$\|Y\|_{\mathcal{K}_{\Phi'}} \leq 2q\, \|\sigma\|_{\delta\mathcal{H}_{\Phi'}} ,$$

further

$$L(X_n) = E(X_n Y_n) .$$

Now by linearity and by the boundedness of the functional

$$|L(X) - L(X_n)| = |L(X - X_n)| \leq B\, \|X - X_n\|_{\mathcal{H}_{\Psi'}} .$$

Since the right-hand side tends to 0 as $n\to+\infty$ it follows that

$$L(X)=\lim_{n\to+\infty}L(X_n)=\lim_{n\to+\infty}E(X_nY_n).$$

This was to be proved.

The proof of this assertion follows in main lines that of GARSIA (see [3], Theorem 1.4.3.). Exceptly the case of the space $\mathcal{H}_1$ treated in Garsia's assertion our result is more general than his one.

REMARK. Let $X\in L_\infty\cap\mathcal{H}_{\varphi'}$. Then under the assuptions of Theorem 2 we have

$$L(X)=E(XY),\quad Y\in\mathcal{K}_{\varphi'}.$$

In fact, the martingale $(Y_n,\mathcal{F}_n)$ constructed in Theorem 2 converges to Y in L_2 as we have shown in Theorem 1. At the same time $(X_n,\mathcal{F}_n)$ is a bounded martingale. Consequently, by Theorem 2

$$L(X)=\lim_{n\to+\infty}E(X_nY_n)=E(XY).$$

It is not difficult to verify that $L_\infty\cap\mathcal{H}_{\varphi'}$ is dense according to the norm of $\mathcal{H}_{\varphi'}$. For this purpose let $\varepsilon>0$ be arbitrary and let $X\in\mathcal{H}_{\varphi'}$ be any random variable. Then there exists an index $n_0=n_0(\varepsilon,X)$ such that we have

$$\|X-X_n\|_{\mathcal{H}_{\varphi'}}\leqq\frac{\varepsilon}{2},$$

if $n\geqq n_0$. On the other hand let X' be a bounded random variable and consider

$$X''=X'-E(X'\mid\mathcal{F}_0).$$

Then trivially, $X''\in\mathcal{H}_{\varphi'}$ since $E(X''\mid\mathcal{F}_0)=0$ and $\sup\limits_{n\geqq 0}|X_n''|$ is bounded (cf. BURKHOLDER, [6], Theorem 15.1.). By the Jensen inequality we further have

$$\|X_{n_0}-X_{n_0}''\|_{\mathcal{H}_{\varphi'}}=\|E(X-X''\mid\mathcal{F}_{n_0})\|_{\mathcal{H}_{\varphi'}}=$$

$$=\left\|\left(\sum_{i=1}^{n_0}\left(E(X-X''\mid\mathcal{F}_i)-E(X-X''\mid\mathcal{F}_{i-1})\right)^2\right)^{1/2}\right\|_{\varphi'}\leqq$$

$$\leqq\left\|\left(2\sum_{i=1}^{n_0}\left(E^2(X-X''\mid\mathcal{F}_i)+E^2(X-X''\mid\mathcal{F}_{i-1})\right)\right)^{1/2}\right\|_{\varphi'}\leqq$$

$$\leqq 2\sqrt{2}\sum_{i=0}^{n_0}\|E(|X-X''|\mid\mathcal{F}_i)\|_{\varphi'}\leqq 2\sqrt{2}(n_0+1)\|X-X''\|_{\varphi'}.$$

Since $E(X\mid\mathcal{F}_0)=0$ a.e. by the Jensen inequality we get

$$\|X-X''\|_{\varphi'}=\|X-X'+E(X'-X\mid\mathcal{F}_0)\|_{\varphi'}\leqq 2\|X-X'\|_{\varphi'}.$$

From this and from the preceding inequality

$$\|X_{n_0}-X_{n_0}''\|_{\mathcal{H}_{\varphi'}}\leqq 4\sqrt{2}(n_0+1)\|X-X'\|_{\varphi'}.$$

The power q being finite the bounded random variables are dense in $L^{\Psi'}$. Consequently, the random variable $X' \in L_\infty$ can be chosen in such a way that

$$4\sqrt{2}\,(n_0+1)\,\|X-X'\|_{\Psi'} \leqq \frac{\varepsilon}{2}$$

hold. It follows that

$$\|X-X''_{n_0}\| \leqq \|X-X_{n_0}\|_{\mathcal{H}_{\Psi'}} + \|X_{n_0}-X''_{n_0}\|_{\mathcal{H}_{\Psi'}} \leqq \varepsilon\,.$$

From these we get the following

Corollary 2. Let $L(X)$ be a bounded linear functional on the elements of $\mathcal{H}_{\Psi'}$ and suppose that q, the power of Ψ', is finite. Then there exists a subset $\mathcal{H}^0_{\Psi'}$ of $\mathcal{H}_{\Psi'}$ such that $\mathcal{H}^0_{\Psi'}$ is dense in $\mathcal{H}_{\Psi'}$ and for any $X \in \mathcal{H}^0_{\Psi'}$ we have

$$L(X) = E(X\,Y)\,,$$

where $Y \in \mathcal{K}_{\Phi'}$.

A similar assertion has been proved by J. O. Strömberg. [5]

4. What is the structure of the $\mathcal{K}_{\Phi'}$-spaces? Theorem 1 characterizes a subset of $\mathcal{K}_{\Phi'}$ when the power q of the conjugate Young-function Ψ' is finite. We shall see in a following paper that under some stronger conditions imposed on Φ' and Ψ' the only elements of $\mathcal{K}_{\Phi'}$ are those which were characterized in Theorem 1.

When Φ' itself has finite power then it is easily seen that $\mathcal{H}_{\Phi'} \subset \mathcal{K}_{\Phi'}$. In fact, if $X \in \mathcal{H}_{\Phi'}$ then by the Burkholder—Davis—Gundy inequality (cf. [6], Theorem 15.1.) it follows that $X^* \in L^{\Phi'}$. Here X^*_n denotes the random variable

$$\sup_{n \geqq 1} |X_n|\,.$$

Consequently, $X^* \in L_2$ and we trivially have

$$E\left((X-X_{n-1})^2 \mid \mathcal{F}_n\right) \leqq E\left(4X^{*2} \mid \mathcal{F}_n\right) \quad \text{a.e.}, \quad n \geqq 1\,.$$

This means that $X \in \mathcal{K}_{\Phi'}$. In the following theorem we prove that this is true also in the opposite sense.

More precisely, we prove the following

Theorem 3. *If the power p of Φ' is finite then $X \in \mathcal{K}_{\Phi'}$ if and only if $X \in \mathcal{H}_{\Phi'}$. More precisely, in this case we have*

$$\|X\|_{\mathcal{K}_{\Phi'}} \leqq \|X\|_{\mathcal{H}_{\Phi'}} \leqq \sqrt{\frac{p}{2}}\,\|X\|_{\mathcal{K}_{\Phi'}}\,.$$

Proof. Suppose that $X \in \mathcal{K}_{\Phi'}$. Let $c > 0$ be an arbitrary constant and consider

$$d_i'^2 = \min(c, d_i^2), \quad i \geqq 1, \quad S_n'^2 = \sum_{i=1}^{n} d_i'^2, \quad n \geqq 1\,,$$

where $d_1, d_2, \ldots$ denotes the differences of the martingale $(X_n, \mathcal{F}_n)$. It is true that $S_n'^2 \in L_\infty$. Let $a > 0$ be another arbitrary constant. Then

$$\Phi'\left(\frac{S'_n}{a}\right)=\sum_{i=1}^{n}\left[\Phi\left(\frac{S'^2_i}{a^2}\right)-\Phi\left(\frac{S'^2_{i-1}}{a^2}\right)\right]\leqq\sum_{i=1}^{n}\varphi\left(\frac{S'^2_i}{a^2}\right)\left(\frac{S'^2_i}{a^2}-\frac{S'^2_{i-1}}{a^2}\right),$$

since for arbitrary $0\leqq x<y$ we have $\varphi(y)-\varphi(x)\leqq\varphi(y)(y-x)$. Introduce then notation

$$\Theta_i=\varphi\left(\frac{S'^2_i}{a^2}\right)-\varphi\left(\frac{S'^2_{i-1}}{a^2}\right),\quad i=1,2,\ldots.$$

Then we have

$$\Phi'\left(\frac{S'_n}{a}\right)\leqq\sum_{i=1}^{n}\sum_{j=1}^{i}\Theta_j\left(\frac{S'^2_i}{a^2}-\frac{S'^2_{i-1}}{a^2}\right)=$$

$$=\sum_{j=1}^{n}\Theta_j\sum_{i=j}^{n}\left(\frac{S'^2_i}{a^2}-\frac{S'^2_{i-1}}{a^2}\right)=\sum_{j=1}^{n}\Theta_j\left(\frac{S'^2_n}{a^2}-\frac{S'^2_{j-1}}{a^2}\right).$$

Consequently, with arbitrary $\gamma\in\Gamma_X^{\Phi'}$ we have

$$E\left(\Phi'\left(\frac{S'_n}{a}\right)\right)\leqq\sum_{j=1}^{n}E\left(\frac{\Theta_j}{a^2}\,E(S'^2_n-S'^2_{j-1}|\mathcal{F}_j)\right)\leqq\sum_{j=1}^{n}E\left(\frac{\Theta_j}{a^2}\,E(\gamma^2|\mathcal{F}_j)\right).$$

The last inequality follows from the fact that

$$0\leqq S'^2_n-S'^2_{j-1}=\sum_{i=j}^{n}d'^2_i\leqq\sum_{i=j}^{n}d^2_i=S^2_n-S^2_{j-1}$$

and from the relation

$$E(S^2_n-S^2_{j-1}|\mathcal{F}_j)=E((X_n-X_{j-1})^2|\mathcal{F}_j)\leqq E(\gamma^2|\mathcal{F}_j).$$

From these we get

$$a^2E\left(\Phi'\left(\frac{S'_n}{a}\right)\right)\leqq b^2E\left(\frac{\gamma^2}{b^2}\varphi\left(\frac{S'^2_n}{a^2}\right)\right).$$

Here $b>0$ is a constant to be determined later. Apply the Young-inequality on the right hand side to obtain

$$a^2E\left(\Phi'\left(\frac{S'_n}{a}\right)\right)\leqq b^2\left[E\left(\Phi\left(\frac{\gamma^2}{a^2}\right)\right)+E\left(\Psi\left(\varphi\left(\frac{S'^2_n}{a^2}\right)\right)\right)\right].$$

Remark that the power p of Φ' is by supposition finite. It follows that the power of Φ is also finite. Namely, it is equal to $\frac{p}{2}$. Consequently,

$$\Psi(\varphi(t))\leqq\left(\frac{p}{2}-1\right)\Phi(t),\quad t\geqq0.$$

From this

$$a^2E\left(\Phi'\left(\frac{S'_n}{a}\right)\right)\leqq b^2\left[E\left(\Phi'\left(\frac{\gamma}{b}\right)\right)+\left(\frac{p}{2}-1\right)E\left(\Phi'\left(\frac{S'_n}{a}\right)\right)\right].$$

Since

$$\Phi'\left(\frac{S_n'}{a}\right)\in L_\infty$$

we get

$$\left[a^2-b^2\left(\frac{p}{2}-1\right)\right]E\left(\Phi'\left(\frac{S_n'}{a}\right)\right)\leqq b^2 E\left(\Phi'\left(\frac{\gamma}{b}\right)\right).$$

Chose $b=\|\gamma\|_{\Phi'}$ and let $a=\sqrt{\frac{p}{2}}\,\|\gamma\|_{\Phi'}$. Then

$$E\left\{\Phi'\left(\frac{S_n'}{\sqrt{\frac{p}{2}}\,\|\gamma\|_{\Phi'}}\right)\right\}\leqq 1\,.$$

If we let $c\uparrow+\infty$ then $S_n'\uparrow S_n$. Φ' being continuous and increasing we obtain

$$\Phi'\left(\frac{S_n'}{\sqrt{\frac{p}{2}}\,\|\gamma\|_{\Phi'}}\right)\uparrow\Phi'\left(\frac{S_n}{\sqrt{\frac{p}{2}}\,\|\gamma\|_{\Phi'}}\right).$$

By the Beppo Levi theorem this implies

$$E\left\{\Phi'\left(\frac{S_n}{\sqrt{\frac{p}{2}}\,\|\gamma\|_{\Phi'}}\right)\right\}\leqq 1$$

and from this we deduce that

$$\|S_n\|_{\Phi'}\leqq\sqrt{\frac{p}{2}}\,\|\gamma\|_{\Phi'}\,.$$

If $n\to+\infty$ then again using the Beppo Levi theorem we see that

$$\|S\|_{\Phi'}\leqq\sqrt{\frac{p}{2}}\,\|X\|_{\mathcal{K}_{\Phi'}}\,,$$

or, in other words

$$\|X\|_{\mathcal{H}_{\Phi'}}\leqq\sqrt{\frac{p}{2}}\,\|X\|_{\mathcal{K}_{\Phi'}}\,.$$

This proves the right-hand side of the inequality. To prove the left one suppose that $X\in\mathcal{H}_{\Phi'}$. Then $E(S^2)$ is trivially finite and the inequality

$$E\left((X-X_{n-1})^2|\mathcal{F}_n\right)=E(S^2-S_{n-1}^2|\mathcal{F}_n)\leqq E(S^2|\mathcal{F}_n)$$

also holds. Consequently, $S\in\Gamma_X^{\Phi'}$ and it follows that $X\in\mathcal{K}_{\Phi'}$. We also have

$$\|X\|_{\mathcal{K}_{\Phi'}}\leqq\|S\|_{\Phi'}=\|X\|_{\mathcal{H}_{\Phi'}}\,.$$

This proves the assertion.

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MAXIMAL INEQUALITIES AND DOOB'S DECOMPOSITION FOR NON-NEGATIVE SUPERMARTINGALES

By

J. MOGYORÓDI

Department of Probability Theory, L. Eötvös University, Budapest

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1. We consider a non-negative supermartingale $(X_n, \mathcal{F}_n)$, $n \geqq 0$, and for the sake of commodity we suppose that $E(X_n) < +\infty$ for all $n \geqq 0$.

The Doob decomposition of this is the following: there exists a unique predictable and increasing sequence $\{A_n\}$, $n \geqq 0$, $A_0 = 0$ a.e. and a unique non-negative martingale $(M_n, \mathcal{F}_n)$, $n \geqq 0$, such that for all $n \geqq 0$ we have

$$X_n = M_n - A_n .$$

The supermartingale $(X_n, \mathcal{F}_n)$ is called potential if $E(X_n) \downarrow 0$, as $n \to +\infty$. Since a non-negative supermartingale has limit a.e., from the preceding assumption we get that the a.e. limit of a potential is equal to 0.

We consider a Young-function $\Phi(x)$ and its conjugate $\Psi(x)$. Together with these we also put

$$\xi(x) = \Psi(\varphi(x)) = x\varphi(x) - \varphi(x) .$$

Here $\varphi(x)$ is the right-hand side derivative of $\Phi(x)$. These are defined for the non-negative values of x. Let

$$p = \sup_{x>0} \frac{x\varphi(x)}{\Phi(x)} .$$

This quantity will be called the power of Φ. We define similarly the power q of Ψ.

Let Φ be a Young-function and denote by $L^\Phi = L^\Phi(\Omega, \mathcal{A}, P)$ the set of those random variables X which are defined on the probability space $(\Omega, \mathcal{A}, P)$ and have the property that there exists at least one positive number a for which

$$E\left(\Phi\left(\frac{|X|}{a}\right)\right) \leqq 1$$

holds. Suppose that $X \in L^\Phi$ and define

$$\|X\|_\Phi = \inf\left(a : a > 0, \ \ E\left(\Phi\left(a^{-1}|X|\right)\right) \leqq 1\right).$$

Then $\|X\|_\Phi$ is a norm on $L^\Phi(\Omega, \mathcal{A}, P)$.

It is well-known that the normed vector space $L^{\Phi}(\Omega, \mathcal{A}, P)$ is complete. This Banach space is called an Orlicz space.

More about the Young-functions and the Orlicz spaces can be found e.g. in [1] or in [2].

For non-negative submartingales $(Z_n, \mathcal{F}_n)$ the following maximal inequalities are true. Suppose that

$$\sup_{n\geq 0} \|Z_n\|_{\Phi} < +\infty .$$

Then

a) the following inequality holds

$$E\left(\xi\left(\frac{\varrho \sup_{n\geq 0} \|Z_n\|_{\Phi}}{Z^*}\right)\right) \leq \frac{1}{\varrho - 1},$$

where $Z^* = \sup_{n\geq 0} Z_n$ and $\varrho > 1$ is an arbitrary constant (cf. [1], Proposition A–3–4);

b) if q, the power of Ψ is finite then

$$\|Z^*\|_{\Phi} \leq q \sup_{n\geq 0} \|Z_n\|_{\Phi} ;$$

more generally,

$$E(\Phi(Z^*)) \leq \sup_{n\geq 0} E(\Phi(q Z_n)),$$

in the sense that if both sides are finite then this inequality holds; if the left-hand side is equal to $+\infty$ then so does the right-hand side (cf. [5], Lemma 1 and [3], Theorem 1);

c) if p, the power of Φ, is finite and

$$c = \sup_{x\geq 0} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t} dt$$

is also finite then

$$\|Z^*\|_{\Phi} \leq pc \sup_{n\geq 0} \|Z_n\|_{\Phi} ;$$

more generally,

$$E(\Phi(Z^*)) \leq \sup_{n\geq 0} E(\Phi(pc Z_n)),$$

in the sense that if both sides are finite then this inequality holds; if the left-hand side is equal to $+\infty$ then so does the right-hand side (cf. [7], Theorem 1.).

In this paper we present maximal inequalities for the non-negative supermartingales and we study the Doob decomposition of these supermartingales.

2. We prove the following

THEOREM 1. *Let $\Phi(x)$ be a Young-function and consider the corresponding function $\xi(x)$. Let $(X_n, \mathcal{F}_n)$, $n \geqq 0$, be a non-negative supermartingale. Denote*

$$X_n^* = \max_{0 \leqq k \leqq n} X_k, \quad n \geqq 0,$$

and let

$$X^* = \sup_{k \geqq 0} X_k.$$

a′) *if*

$$0 < \|M_n\|_\Phi < +\infty, \quad n \geqq 0,$$

where

$$X_n = M_n - A_n, \quad n \geqq 0,$$

is the Doob decomposition of X_n then with arbitrary constant $\varrho > 1$ we have

$$E\left(\xi\left(\frac{X_n^*}{\varrho \|M_n\|_\Phi}\right)\right) \leqq \frac{1}{\varrho - 1}.$$

b′) *If the martingale $(M_n, \mathcal{F}_n)$ is regular, i.e.*

$$M_n = E(M_\infty | \mathcal{F}_n), \quad n \geqq 0,$$

where

$$M_\infty = \lim_{n \to +\infty} \text{ a.e. } M_n = X_\infty + A_\infty$$

and

$$X_\infty = \lim_{n \to +\infty} \text{ a.e. } X_n; \quad A_\infty = \lim_{n \to +\infty} \text{ a.e. } A_n,$$

further, if $0 < \|X_\infty\|_\Phi < +\infty$ and $0 < \|A_\infty\|_\Phi < +\infty$ then

$$E\left(\xi\left(\frac{X^*}{\varrho\left(\sup_{n \geqq 0} \|X_n\|_\Phi + \|A_\infty\|_\Phi\right)}\right)\right) \leqq \frac{1}{\varrho - 1}.$$

c′) *If, in addition, $(X_n, \mathcal{F}_n)$ is potential and $\|A_\infty\|_\Phi < +\infty$ then*

$$E\left(\xi\left(\frac{X^*}{\varrho \|A_\infty\|_\Phi}\right)\right) \leqq \frac{1}{\varrho - 1}.$$

d′) *If the power q of Ψ, the conjugate Φ, is finite then we have*

$$E(\Phi(X^*)) \leqq \sup_{n \geqq 0} E\big(\Phi(q(X_n + A_n))\big)$$

in the sense of b) *of section* 1.

e′) *Finally, if the power p of Φ as well as the quantity*

$$c = \sup_{x > 0} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t} dt$$

are finite then

$$E(\Phi(X^*)) \leq \sup_{n\geq 0} E\Big(\Phi\big(pc\,(X_n+A_n)\big)\Big)$$

in these sense of c) *of section* 1.

Proof. a′) Since for arbitrary $k\geq 0$ we have

$$X_k \leq M_k\,,$$

it follows that

$$X_n^* \leq M_n^*, \quad n\geq 0\,.$$

Thus, $\xi(x)$ being an increasing function, we get from a) of section 1 that

$$E\left(\xi\left(\frac{X_n^*}{\varrho\,\|M_n\|_\Phi}\right)\right) \leq E\left(\xi\left(\frac{M_n^*}{\varrho\,\|M_n\|_\Phi}\right)\right) \leq \frac{1}{\varrho-1}\,.$$

b′) If $(M_n, \mathcal{F}_n)$ is regular, consequently, of the form

$$M_n = E(M_\infty|\mathcal{F}_n) = E(X_\infty|\mathcal{F}_n) + E(A_\infty|\mathcal{F}_n)\,,$$

then again by a) of section 1 we have

$$E\left(\xi\left(\frac{X^*}{\varrho\,(\sup\limits_{n\geq 0}\|X_n\|_\Phi + \|A_\infty\|_\Phi)}\right)\right) = E\left(\xi\left(\frac{X^*}{\varrho\,(\sup\limits_{n\geq 0}\|X_n\|_\Phi + \sup\limits_{n\geq 0}\|A_n\|_\Phi)}\right)\right) \leq$$

$$\leq E\left(\xi\left(\frac{X^*}{\varrho\sup\limits_{n\geq 0}\|X_n+A_n\|_\Phi}\right)\right) \leq E\left(\xi\left(\frac{M^*}{\varrho\sup\limits_{n\geq 0}\|M_n\|_\Phi}\right)\right) \leq \frac{1}{\varrho-1}\,,$$

since trivially

$$\sup_{n\geq 0}\|M_n\|_\Phi = \sup_{n\geq 0}\|X_n+A_n\|_\Phi \leq \sup_{n\geq 0}\|X_n\|_\Phi + \sup_{n\geq 0}\|A_n\|_\Phi =$$

$$= \sup_{n\geq 0}\|X_n\|_\Phi + \|A_\infty\|_\Phi\,.$$

Here, as usual,

$$M^* = \sup_{n\geq 0} M_n\,.$$

c′) If $(X_n, \mathcal{F}_n)$, $n\geq 0$, is potential then $X_\infty = 0$ a.e. and from the preceding inequality

$$E\left(\xi\left(\frac{X^*}{\varrho\,\|A_\infty\|_\Phi}\right)\right) \leq \frac{1}{\varrho-1}\,,$$

since in this case

$$M_n = E(A_\infty|\mathcal{F}_n)$$

and

$$\sup_{n\geq 0}\|M_n\|_\Phi \leq \|A_\infty\|_\Phi\,.$$

d′) Suppose now that q, the power of Ψ, is finite. Then, by b) of section 1 and by the fact that $X_n^* \leqq M_n^*$ we get

$$E(\Phi(X^*)) \leqq E(\Phi(M^*)) \leqq \sup_{n \geqq 0} E(\Phi(q M_n)) = \sup_{n \geqq 0} E\Big(\Phi\big(q(X_n + A_n)\big)\Big).$$

e′) Finally, suppose that the power of p of Φ as well as the quantity

$$c = \sup_{x>0} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t}\, dt$$

are finite. Then, by c) of section 1 we get

$$E(\Phi(X^*)) \leqq E(\Phi(M^*) \leqq \sup_{n \geqq 0} E(\Phi(pc M_n) =$$

$$= \sup_{n \geqq 0} E\Big(\Phi\big(pc(X_n + A_n)\big)\Big).$$

This proves the assertion.

Remarks. (a) Suppose that q, the power of Ψ, is finite. Then the inequality

$$\Psi(\varphi(x)) = \xi(x) = x\varphi(x) - \Phi(x) \geqq \frac{1}{q-1}\Phi(x)$$

is trivially valid. If $(X_n, \mathcal{F}_n)$ is potential then by choosing $\varrho = q$ we get from assertion c′) of the preceding theorem that

$$\frac{1}{q-1} E\left(\Phi\left(\frac{X^*}{q\|A_\infty\|_\Phi}\right)\right) \leqq E\left(\xi\left(\frac{X^*}{q\|A_\infty\|_\Phi}\right)\right) \leqq \frac{1}{q-1},$$

provided that $\|A_\infty\|_\Phi < +\infty$. This shows that $\|X^*\|_\Phi \leqq q\|A_\infty\|_\Phi$. This inequality has been obtained by Neveu [1]. Proposition VIII−1−4, in the case of the Young-function $\Phi(x) = x^p/p$. This inequality could also be obtained from assertion d′) of the theorem.

(b) Suppose that the power p of Φ is finite and that the quantity

$$c = \sup_{x>0} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t}\, dt$$

is also finite. If $(X_n, \mathcal{F}_n)$ is potential, therefore necessarily of the form

$$X_n = E(A_\infty - A_n | \mathcal{F}_n) = E(A_\infty | \mathcal{F}_n) - A_n,$$

then from assertion e′) of the preceding theorem we get

$$E(\Phi(X^*)) \leqq \sup_{n \geqq 0} E\Big(\Phi\big(pc\, E(A_\infty | \mathcal{F}_n)\big)\Big).$$

From this, if $A_\infty \in L^\Phi$, by the Jensen inequality we obtain

$$E\left(\Phi\left(\frac{X^*}{pc\,\|A_\infty\|_\Phi}\right)\right) \leq \sup_{n\geq 0} E\left(\Phi\left(E\left(\frac{A_\infty}{\|A_\infty\|_\Phi}\,\Big|\,\mathcal{F}_n\right)\right)\right) \leq$$

$$\leq E\left(\Phi\left(\frac{A_\infty}{\|A_\infty\|_\Phi}\right)\right) \leq 1\,,$$

which shows that

$$\|X^*\|_\Phi \leq pc\,\|A_\infty\|_\Phi\,.$$

This is also the generalization of the inequality of Neveu.

3. Now we turn to the study of the deceomposition of Doob.

THERed 2. *Let $(X_n, \mathcal{F}_n)$, $n \geq 0$, be a non-negative supermartingale and consider its Doob decomposition*

$$X_n = M_n - A_n, \quad n \geq 0\,.$$

Let Φ be a Young-function. Suppose that

a) *q, the power of Ψ, is finite, or that*

b) *the power p of Φ is finite together with the quantity*

$$c = \sup_{x>0} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t}\,dt.$$

Then under one of these conditions the assumption

$$\sup_{n\geq 0} \|M_n\|_\Phi < +\infty$$

is equivalent to the conjunction of the two conditions

$$\sup_{n\geq 0} \|X_n\|_\Phi < +\infty \quad \text{and} \quad \|A_\infty\|_\Phi < +\infty\,.$$

In case b) *X_n converges in L^Φ-norm to its a.e. limit X_∞.*

PROOF. Since

$$M_n = X_n + A_n\,,$$

we get without any supposition concerning the powers of Φ and of Ψ that

$$\sup_{n\geq 0} \|M_n\|_\Phi \leq \sup_{n\geq 0} \|X_n\|_\Phi + \|A_\infty\|_\Phi\,.$$

Consequently, if the right-hand side is finite then so does the left one.

Now we prove the converse assertion. To this end suppose first the validity of condition a). Since

$$X^* \leq M^*\,,$$

and since

$$\|M^*\|_\Phi \leq q \sup_{n\geq 0} \|M_n\|_\Phi\,,$$

we get that

$$\|X_n\|_\Phi \leqq \|X_n^*\|_\Phi \leqq \|X^*\|_\Phi \leqq q \sup_{n\geqq 0} \|M_n\|_\Phi .$$

Therefore,

$$\sup_{n\geqq 0} \|X_n\|_\Phi \leqq \|X^*\|_\Phi \leqq q \sup_{n\geqq 0} \|M_n\|_\Phi .$$

Further, since

$$A_n = M_n - X_n ,$$

we see that

$$\|A_n\|_\Phi \leqq \|M_n\|_\Phi + \|X_n\|_\Phi \leqq (q+1) \sup_{n\geqq 0} \|M_n\|_\Phi .$$

From this

$$\|A_\infty\|_\Phi = \sup_{n\geqq 0} \|A_n\|_\Phi \leqq (q+1) \sup_{n\geqq 0} \|M_n\|_\Phi .$$

Suppose now the validity of b). Then by the same way as in the preceding step we get

$$\|M^*\|_\Phi \leqq pc \sup_{n\geqq 0} \|M_n\|_\Phi$$

and we deduce that

$$\|X^*\|_\Phi \leqq \|M^*\|_\Phi \leqq pc \sup_{n\geqq 0} \|M_n\|_\Phi .$$

From this it follows as before that

$$\sup_{n\geqq 0} \|X_n\|_\Phi \leqq pc \sup_{n\geqq 0} \|M_n\|_\Phi .$$

Further, since

$$A_n = M_n - X_n$$

we obtain

$$\|A_\infty\|_\Phi = \sup_{n\geqq 0} \|A_n\|_\Phi \leqq \sup_{n\geqq 0} \|M_n\|_\Phi + \sup_{n\geqq 0} \|X_n\|_\Phi \leqq (pc+1) \sup_{n\geqq 0} \|M_n\|_\Phi .$$

In either case the assumption

$$\sup_{n\geqq 0} \|M_n\|_\Phi < +\infty$$

implies the finiteness of the quantities

$$\sup_{n\geqq 0} \|X_n\|_\Phi \quad \text{and} \quad \|A_\infty\|_\Phi .$$

Finally, in case b) we have

$$X_\infty - X_n = M_\infty - M_n - (A_\infty - A_n) .$$

From this

$$\|X_\infty - X_n\|_\Phi \leqq \|M_\infty - M_n\|_\Phi + \|A_\infty - A_n\|_\Phi .$$

By the supposition that p is finite it follows that (cf. [6], Theorem I.)

$$\|M_\infty - M_n\|_\Phi \to 0 ,$$

while for arbitrary Φ we trivially have that

$$\|A_\infty - A_n\|_\Phi \to 0, \quad \text{if} \quad A_\infty \in L^\Phi .$$

This proves the assertion.

4. It seems to be interesting to prove maximal inequalities for concave Young-functions of non-negative supermartingales as well. The concave Young-functions have the general form

$$\Phi(x) = \int_0^x \varphi(v)\, dv ,$$

where $\varphi(v)$ is a non-negative decreasing and right continuous function such that the above integral is finite for arbitrary $x>0$. It can be easily seen that $\Phi(x)$ is really concave and trivially we have for arbitrary $x_0>0$

$$\Phi(x) \leqq \Phi(x_0) + \varphi(x_0)(x-x_0)^+ .$$

This implies that for any non-negative supermartingale $(X_n, \mathcal{F}_n)$, such that $E(X_n) < +\infty$ for all $n \geqq 0$, we have

$$E(\Phi(X_n)) < +\infty .$$

For arbitrary $x \geqq 0$ we also consider the non-negative function

$$\xi(x) = \Phi(x) - x\varphi(x) .$$

First we prove the following

THEOREM 3. *Let Φ be a concave Young-function and let $(X_n, \mathcal{F}_n)$, $n \geqq 0$, be a non-negative supermartingale. Then*

$$E(\xi(X^*)) \leqq E(\Phi(X_0)) .$$

PROOF. Since $\xi(x)$ is non-negative and Φ is continuous we see that

$$\lim_{x\to 0} x\varphi(x) = 0 .$$

Let us consider the maximal inequality

$$xP(X^* \geqq x) \leqq E(\min(X_0, x))$$

which is valid for the non-negative supermartingale $(X_n, \mathcal{F}_n)$ (cf. NEVEU [1], Proposition II-2-7). Integrate this inequality on the interval $(0, +\infty)$ with respect to the measure generated by the right continuous and increasing function $-\varphi(x)$. By the Fubini theorem we then get

$$E\left(\int_0^{+\infty} \chi(X^* \geqq x)\, d(-\varphi(x))\right) \leqq E\left(\int_0^{+\infty} \min(X_0, x)\, d(-\varphi(x))\right).$$

Noting that for arbitrary $z \geqq 0$ we have

$$\int_0^z x\, d(-\varphi(x)) = \xi(z) ,$$

we obtain from the preceding inequality

$$E(\xi(X^*)) \leqq E\left[\xi(X_0) + X_0 \int_{X_0}^{+\infty} d(-\varphi(x))\right] =$$

$$= E\Big(\xi(X_0) + X_0(\varphi(X_0)) - \varphi(+\infty)\Big) \leqq E(\xi(X_0) + X_0 \varphi(X_0)) = E(\Phi(X_0)),$$

and this proves the theorem.

REMARK. Let

$$r = \sup_{x>0} \frac{x\varphi(x)}{\Phi(x)} .$$

Since $\Phi(x)$ cannot vanish for $x>0$ (exceptly the trivial case of $\Phi(x)=0$) we see that r exists and is not greater than 1. Suppose that $r<1$. In this case

$$\xi(x) = \Phi(x) - x\varphi(x) = \Phi(x)\left(1 - \frac{x\varphi(x)}{\Phi(x)}\right) \geqq \Phi(x)(1-r)$$

and the inequality of the preceding theorem gives that

$$E(\Phi(X^*)) \leqq \frac{1}{1-r} E(\Phi(X_0)).$$

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GROUPS WITHOUT CERTAIN SUBGROUPS FORM A FITTING CLASS

By

PÉTER HERMANN

Department of Algebra and Number Theory, L. Eötvös University, Budapest

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NOTATION. Finite one step not p-nilpotent (i.e. with all proper subgroups p-nilpotent) groups of order $p^m \cdot q^n$ (for any m, n) will be referred to as (p, q)-groups. For different primes p and q let $\mathcal{N}(p, q)$ denote the set of finite groups without (p, q)-subgroups.

The basic properties of (p, q)-groups were given by N. ITO [3] (see also [2]). We shall use the fact that every (p, q)-group possesses a normal Sylow p-subgroup. As $\mathcal{N}(p) = \bigcap_q \mathcal{N}(p, q)$ is the class of finite p-nilpotent groups, $\mathcal{N}(p, q)$ may be considered as a localization of p-nilpotence. Our aim is to show that some of the well-known "nice" properties of $\mathcal{N}(p)$ remain valid for $\mathcal{N}(p, q)$, too. Before doing so we remark: A finite group G belongs to $\mathcal{N}(p, q)$ iff for any p-subgroup P and q-subgroup Q, $N_Q(P) = C_Q(P)$ holds.

PROPOSITION 1.

(1) If $G \in \mathcal{N}(p, q)$ and $H \leq G$ then $H \in \mathcal{N}(p, q)$
(2) If $G_1, G_2 \in \mathcal{N}(p, q)$ then $G_1 \times G_2 \in \mathcal{N}(p, q)$
(3) If $G \in \mathcal{N}(p, q)$ and $H \triangleleft G$ then $G/H \in \mathcal{N}(p, q)$

REMARK. (1) and (2) are obvious. (3) is an unpublished result of K. CORRÁDI.

PROOF of (3). Assume it were false, and let G be a minimal counterexample. We denote by R a Sylow r-subgroup of H then $G = H \cdot N_G(R)$. As $N_G(R)/N_H(R) \simeq G/H$, $N_G(R) = G$ by the minimality of G, thus H is nilpotent. Also by the minimality of G, $H = R$ is elementary abelian and G/H is a (p, q)-group. Suppose that $p \neq r \neq q$ then by the Schur—Zassenhaus theorem G contains a subgroup isomorphic to G/H, a contradiction. If $r = p$, G would be a $\{p, q\}$-group with a normal Sylow p-subgroup not centralized by a q-subgroup, contradicting to $G \in \mathcal{N}(p, q)$. Thus $r = q$ and G is p-nilpotent by Ito's theorem [3], contrary to $G/H \notin \mathcal{N}(p, q)$. ▌

COROLLARY. $\mathcal{N}(p, q)$ is a formation. ▌

EXAMPLE. Let $\hat{G}$ denote the group of automorphisms of the followin, graph:

Let G be the group of automorphisms in $\hat{G}$ inducing even permutations on the set of the edges. (G is clearly the wreath product $S_2 \wr A_6$). We denote by H the subgroup consisting of the elements fixing all edges. Let Φ be the set of even permutations in H and Z be the two-element subgroup of H generated by its unique fixed-point-free element; then $\Phi \lhd G$ and $Z \leq Z(G)$. Φ and Z are the only nontrivial normal subgroups of G in H as G is 4-transitive on the edges. Suppose that $\Phi \not\leq L$ for some maximal subgroup L; $G = H \cdot L$ and H is abelian, hence $H \cap L \lhd G$, thus $H \cap L = Z$ or $H \cap L = 1$; in both cases $H \cap L = \Phi \cap L$ which is impossible because of the assumption $H \cdot L = G = \Phi \cdot L$. So Φ is contained in the Frattini-subgroup $\Phi(G)$ of G. Clearly $H \neq \Phi(G)$, hence $\Phi = \Phi(G)$ as $G/H \simeq A_6$ is simple. $|(G/\Phi)/(H/\Phi)| = 2^3 \cdot 3^2 \cdot 5$ forces $(G/\Phi)/(H/\Phi) \in \mathcal{K}(2, 5)$ hence $G/\Phi \in \mathcal{K}(2, 5)$ as $H/\Phi \leq Z(G/\Phi)$. On the other hand $G \notin \mathcal{K}(2, 5)$ because the normal 2-subgroup H is not centralized by the 5-subgroups. Thus we have proved

PROPOSITION 2. The formations $\mathcal{K}(p, q)$ are not necessarily saturated. ∎

THEOREM. *$\mathcal{K}(p, q)$ is a Fitting-class.*

REMARK. It was proved in [1] that the p-solvable part of $\mathcal{K}(p, q)$ is a Fitting-class.

PROOF. We have only to prove that whenever G is a finite group, $N \lhd G$, $M \lhd G$ and $N, M \in \mathcal{K}(p, q)$ then $\langle N, M\rangle \in \mathcal{K}(p, q)$ also holds. If G is a minimal counterexample to this statement then $G = N \cdot M$ and G contains a (p, q)-subgroup U. Let U_p denote the unique Sylow p-subgroup of U. Let $H = \langle N, U\rangle$ then $H = N(H \cap M) = G$ by the minimality of G. Let $N_1 \geq N$ and N_1 be a maximal normal subgroup in G then $N_1 = N(N_1 \cap M) \in \mathcal{K}(p, q)$ also by the minimality of G. Thus we may assume that N and M are maximal normal subgroups of G hence (because of $\langle N, U\rangle = G = \langle M, U\rangle$) $U_p < N \cap M$. Let $U_p \leq P$ where P is a Sylow p-subgroup of $N \cap M$, then $M = (N \cap M) N_M(P)$ hence $G = N \cdot N_M(P)$. We denote by Q a Sylow q-subgroup of $N_M(P)$; as $|G:N| = q$, $G = N \cdot Q$; by $Q \leq C_M(P) \leq C_M(U_p)$ this yields $G = N \cdot C_M(U_p)$, therefore $N_G(U_p) = N_N(U_p) \cdot C_M(U_p)$. Let Q_1 denote a Sylow q-subgroup of $N_G(U_p)$ containing a Sylow q-subgroup U_q of U then (as $N_N(U_p)$ and $C_N(U_p)$ are normal in $N_G(U_p)$)

$$U_q \leq Q_1 = (Q_1 \cap N_N(U_p)) \cdot (Q_1 \cap C_M(U_p)) < C_G(U_p),$$

a contradiction. ∎

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THE FARTHEST POINT PROBLEM IN SLIGHTLY PERTURBED NORM

By

ADAM P. BOSZNAY

Department of Mathematics, Budapest University of Technology,
Faculty of Mechanical Engineering, Budapest

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Introduction

Let us recall the definition of the so-called farthest point map. Let $(X, \|\cdot\|)$ be a real normed linear space, $K \subset X$ a bounded set. The mapping $Q_K : X \to 2^K$ defined by

$$Q_K(x) = \{k \in K: \ \|x-k\| = \sup_{k' \in K} \|x-k'\|\}$$

is called the farthest point map. The following question naturally arises: does there exist a non-singleton K with the property that for every $x \in X$ the set $Q_K(x)$ is a singleton?

This question — as far as we know — is not solved in general. There are many special cases in which the answer is negative: for example in finite dimensional spaces [1], in the case of norm-compact K, in the space c_0 [2]. We proved in an earlier paper [3] that for any normed linear space there exists an equivalent renorming of the space such that in the new norm the answer is negative. In this paper we construct for every $\varepsilon > 0$ a norm $\|\cdot\|_1$ such that in this norm the answer is negative, and

$$1-\varepsilon < \frac{\|x\|_1}{\|x\|} \leq 1+\varepsilon$$

for all $0 \neq x \in X$.

The result

We say the set K is uniquely remotal if for every $x \in X$ the set $Q_K(x)$ contains exactly one element. In this case we denote also this element by $Q_K(x)$.

THEOREM. *Let $(X, \|\cdot\|)$ be a real normed linear space, $\varepsilon > 0$ an arbitrary fixed number. Then there exists a norm $\|\cdot\|_1$ on X such that*

$$(1) \qquad 1-\varepsilon \leq \frac{\|x\|_1}{\|x\|} \leq 1+\varepsilon \quad \textit{for all} \quad x \in X, \quad x \neq 0,$$

and that in $(X, \|\cdot\|_1)$ all uniquely remotal sets are singletons.

PROOF. We distinguish between two cases.

Case 1. dim $X<2$. This case is trivial.

Case 2. dim $X\geq 2$. In the sequel we deal with this case.

Let $Y\subset X$ be a two-dimensional subspace. Clearly, there exist $n\in\mathbf{N}$ and $y_{\pm 1}, y_{\pm 2}, \dots, y_{\pm n}\in Y$ with the following properties:

(i) $y_i=-y_{-i}$ for $i=1,2,\dots,n$;

(ii) $\|y_i\|=1$ for $i=1,2,\dots,n$;

(iii) there exist $x^*_{\pm 1}, x^*_{\pm 2}, \dots, x^*_{\pm n}\in X^*$

such that

$$x^*_i=-x^*_{-i},\quad \|x^*_i\|=1,$$

x^*_i is a support functional of the unit sphere $S_{\|\cdot\|}(0,1)$ at y_i, and the norm $\|\cdot\|_1'$ on Y defined by

$$\|y\|_1'=\max_{1\leq i\leq n}|x^*_{\pm i}(y)|$$

satisfies

$$\left(1+\frac{\varepsilon}{8}\right)\|y\|\geq\|y\|_1'\geq\left(1-\frac{\varepsilon}{2}\right)\|y\| \tag{2}$$

for all $y\in Y$.

Let us now construct the norm $\|\cdot\|_1$. We define

$$\|x\|_1=\max\left\{\|x\|,\ \max_{1\leq i\leq n}|x^*_{\pm i}(x)|\left(1+\frac{3\varepsilon}{4}\right)\right\}.$$

Using (2) we have (1) for sufficiently small $\varepsilon>0$. (2) and an easy compactness argument imply that there exists $1>\delta>0$ with the property the every $x\in X$ which fulfils

$$\inf\{\|x-y\|;\ y\in Y,\ \|y\|\geq 1\}<\delta \tag{3}$$

we have

$$\left(1+\frac{3\varepsilon}{4}\right)\max_{1\leq i\leq n}|x^*_{\pm i}(x)|>\|x\|. \tag{4}$$

We prove now that all uniquely remotal sets in $(X,\|\cdot\|_1)$ are singletons. We show this in an indirect way.

Let $K\subset X$ be uniquely remotal in the norm $\|\cdot\|_1$. Introducing

$$d=\sup_{k\in K}\|k\|$$

it is easy to prove, using (3) and (4), that for all y belonging to the set

$$H\equiv\left\{y\in X;\ \exists y'\in Y,\ \|y'\|\geq\frac{2d}{\delta},\ \|y-y'\|\leq d\right\} \tag{5}$$

we have

$$\|y-k\|_1 = \left(1+\frac{3\varepsilon}{4}\right) \max_{1\le i\le n} |x_i^*(y-k)|$$

for all $k\in K$. Applying this to $k = Q_K(y)$ we obtain

$$\|y-Q_K(y)\|_1 = \left(1+\frac{3\varepsilon}{4}\right) \max_{1\le i\le n} |x_i^*(y-Q_K(y))| \quad \text{for} \quad y\in H. \tag{6}$$

Now, modifying an idea of ASPLUND [2], we shall prove that the restriction of the mapping Q_K to H is constant. First we prove that if $z_1, z_2\in H$ (see (5)), and

$$\|z_1-Q_K(z_1)\|_1 = \left(1+\frac{3\varepsilon}{4}\right) x_i^*(z_1-Q_K(z_1)),$$

$$\|z_2-Q_K(z_2)\|_1 = \left(1+\frac{3\varepsilon}{4}\right) x_i^*(z_2-Q_K(z_2))$$

for some $1\le i\le n$ (see (6)), then

$$Q_K(z_1) = Q_K(z_2). \tag{7}$$

We may assume without loss of generality that

$$x_i^*(Q_K(z_1)) \le x_i^*(Q_K(z_2)).$$

Using this and the definition of $\|\cdot\|_1$ we obtain

$$\|z_2-Q_K(z_2)\|_1 = \left(1+\frac{3\varepsilon}{4}\right) x_i^*(z_2-Q_K(z_2)) \le$$

$$\le \left(1+\frac{3\varepsilon}{4}\right) x_i^*(z_2-Q_K(z_1)) \le \|z_2-Q_K(z_1)\|_1,$$

which implies (7).

We can similarly prove that if $z_1', z_2'\in H$ and

$$\|z_1'-Q_K(z_1')\|_1 = \left(1+\frac{3\varepsilon}{4}\right) x_{-i}^*(z_1'-Q_K(z_1')),$$

$$\|z_2'-Q_K(z_2')\|_1 = \left(1+\frac{3\varepsilon}{4}\right) x_{-i}^*(z_2'-Q_K(z_2'))$$

for some $1\le i\le n$, then

$$Q_K(z_1') = Q_K(z_2'). \tag{8}$$

(7) and (8) imply that the set $Q_K(H)$ is finite. Using now the elementary fact that $Q_K^{-1}(k)$ is closed for each $k\in K$ (this follows from the definition of Q_K), we obtain that H is the union of finitely many pairwise disjoint closed

sets (in the relative topology of H). On the other hand, because of $\dim Y = 2$, H is connected. So there exists a $k^* \in K$ with the property

$$Q_K(y) = k^* \quad \text{for all} \quad y \in H, \tag{9}$$

as we have claimed.

By the definition of H, it is easy to prove that there exists an element x^* such that $k^*+x^* \in H$ and $k^*-x^* \in H$. By (9),

$$Q_K(k^*+x^*) = Q_K(k^*-x^*) = k^*. \tag{10}$$

This implies

$$K \subset S_{\|\cdot\|_1}(k^*+x^*, \ \|x^*\|_1) \cap S_{\|\cdot\|_1}(k^*-x^*, \ \|x^*\|_1).$$

Using the triangle inequality, an easy computation shows that

$$\operatorname{int} S_{\|\cdot\|_1}(k^*+x^*, \ \|x^*\|_1) \cap S_{\|\cdot\|_1}(k^*-x^*, \ \|x^*\|_1) = \emptyset$$

and

$$S_{\|\cdot\|_1}(k^*+x^*, \ \|x^*\|_1) \cap \operatorname{int} S_{\|\cdot\|_1}(k^*-x^*, \ \|x^*\|_1) = \emptyset.$$

From these formulas

$$K \subset \{x \in K; \ \|k^*+x^*-x\|_1 = \|x^*\|_1, \ \|k^*-x^*-x\|_1 = \|x^*\|_1\}.$$

Here, obviously, $x = k^*$ is an element of the right-hand side and, in view of (10), the only element of it. ∎

I should like to thank J. Bognár and L. Stacho for useful advice.

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A REMARK ON A PROBLEM OF G. GODINI

By

ADAM P. BOSZNAY

Department of Mathematics, Budapest University of Technology,
Faculty of Mechanical Engineering, Budapest

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Let G be a linear subspace of the normed linear space $(X, \|\cdot\|)$. The metric projection $P_G : X \to 2^G$ is defined by the relation

$$P_G(x) = \left\{ y \in G;\ \|x - y\| = \inf_{y' \in G} \|x - y'\| \right\}.$$

Considering the special case $P_G(x) \neq 0$ for any $x \in X$, we can define

$$\Omega_G(x, \varepsilon) = \sup \{ \Delta(P_G(x), P_G(y));\ y \in X,\ \|y - x\| < \varepsilon \},$$

where $\Delta(P_G(x), P_G(y))$ denotes the Hausdorff distance of $P_G(x)$ and $P_G(y)$. Further we can define

$$\Omega_G(X, \varepsilon) = \sup_{x \in X} \Omega_G(x, \varepsilon)$$

and

$$\Omega_n(X, \varepsilon) = \sup_{\dim G = n} \Omega_G(X, \varepsilon).$$

In [1], G. Godini raised the following question: Does there exist an $n \in \mathbf{N}$ such that

$$\Omega_n(X, 1) < \infty$$

for all normed linear spaces $(X, \|\cdot\|)$? In this paper we give a negative answer to this question.

The result

Proposition. *There exists an infinite dimensional normed linear space $(X, \|\cdot\|)$ such that for all $n \in \mathbf{N}$*

$$\Omega_n(X, 1) = \infty.$$

PROOF. We shall construct normed linear spaces $(X_n, \|\cdot\|_n)$ so that there exists an n-dimensional Chebyshev subspace $G_n \subset X_n$ (i.e. a subspace G_n of $(X_n, \|\cdot\|_n)$ such that $P_G(x)$ is a singleton for all $x \in X$) with the property

$$\Omega_{G_n}(X_n, 1) = \infty .$$

After this, we can construct $(X, \|\cdot\|)$ in the following way:

$$X = \left\{(x_1, x_2, \ldots, \ldots);\ x_i \in X_i, \sum_{i=1}^{\infty} \|x_i\|_i^2 < \infty\right\},$$

$$\|(x_1, x_2, \ldots, \ldots)\| = \sqrt{\sum_{i=1}^{\infty} \|x_i\|_i^2}\ .$$

Construction of X_n, $\|\cdot\|_n$, and G_n. Let $X_n = \mathbf{R}^{n+2}$ and

$$G_n = \{(\alpha_1, \ldots, \alpha_n, 0, 0);\ \alpha_i \in \mathbf{R}\}.$$

Let us denote by $\|\cdot\|_0$ the euclidean norm of X_n.

After defining the subspace

$$H_n = \{(0, 0, \ldots, 0, \beta_1, \beta_2);\ \beta_1, \beta_2 \in \mathbf{R}\},$$

introduce

$$S_n = \{x \in H_n;\ \|x\|_0 = 1\}.$$

Clearly, S_n is isometric to the unit circle. For every $x \in S_n$ we denote by $\operatorname{arc}(x)$ the length of the shorter arc in S_n between x and $(0, 0, \ldots, 0, 1)$.

Defining now the set

$$S_n' = \left\{y;\ y = x + \frac{1}{10}\operatorname{sgn}\left(\operatorname{arc}(x) - \frac{\pi}{2}\right)\sqrt{\left|\operatorname{arc}(x) - \frac{\pi}{2}\right|}\cdot(1, 0, \ldots, 0)\right.$$
$$\left.\text{for some}\quad x \in S_n\right\}$$

for all $y \in S_n'$ we clearly have $-y \in S_n'$ too. This implies that the Minkowski functional of

$$co\left(S_n' \cup \left\{x \in X_n;\ \|x\|_0 \le \frac{9}{10}\right\}\right)$$

is a norm. We denote this norm by $\|\cdot\|_n$.

We can easily deduce that in the norm $\|\cdot\|_n$, G_n is a Chebyshev subspace, and if

$$x = (\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \alpha_{n+2}) \in X_n \setminus G_n$$

then

$$P_{G_n}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \alpha_{n+2}) = (t, \alpha_2, \ldots, \alpha_n, 0, 0),$$

where

$$t = \alpha_1 + \inf_{g \in G_n} \|x-g\|_n \cdot \sqrt{\operatorname{arc}\left(\frac{(0, \ldots, 0, \alpha_{n+1}, \alpha_{n+2})}{\inf_{g \in G_n} \|x-g\|_n}\right) - \frac{\pi}{2}} \cdot$$

$$\cdot \frac{1}{10} \cdot \operatorname{sgn}\left(\operatorname{arc}\left(\frac{(0, \ldots, 0, \alpha_{n+1}, \alpha_{n+2})}{\inf_{g \in G_n} \|x-g\|_n}\right) - \frac{\pi}{2}\right).$$

The latter formula shows that for any $c>0$ we can choose $y_1, y_2 \in X_n$ so that $\|y_1 - y_2\|_n \leq 1$ and $\|P_{G_n}(y_1) - P_{G_n}(y_2)\| > c$. This implies $\Omega_n(X_n, 1) = \infty$. ∎

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A NOTE ON MY PAPER "UNIQUE PRIME FACTORIZATION IN IMAGINARY QUADRATIC NUMBER FIELDS"

By

EDIT GYARMATI

Department of Algebra and Number Theory, L. Eötvös Univesity, Budapest

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Let $E(d)$ denote the ring of integers of the algebraic number field $Q(\sqrt{d})$ (the extension of the rational field by $\sqrt{d}$), where d is a rational integer.

It was an open question for a long time, which are those $d<0$ values for which $E(d)$ is a unique factorization domain (UFD). It was proved by non-elementary methods that at most ten such $d<0$ exist and nine of them are

$$d=-3,\ -4,\ -7,\ -8,\ -11,\ -19,\ -43,\ -67 \quad \text{and} \quad -163\,.$$

In [1] I gave an elementary proof by a unified method using geometry of numbers that $E(d)$ is UFD for these values of d, and I obtained some information for the possible tenth value of d too. Later H. M. STARK gave a non-elementary proof in [4] that no such tenth $d<0$ existed.

In my proof I had to face several technical difficulties, because I was not aware of the following simple fact:

If the norm of an $E(d)$-integer is a rational prime, then this integer is not only irreducible, but is also prime in $E(d)$.*

This statement can be easily proved by ideal theory. An elementary but complicated proof was given in [3]. For a simple elementary proof see Lemma 4 in [5].

Using this proposition my proof can be significantly simplified and shortened which I shall sketch below. For further generalizations and comments see [5] and [6].

My proof was based on a generalized Zermelo-type argument. Assuming that for some α in $E(d)$

$$(1) \qquad \alpha=\pi_1\pi_2\ldots\pi_r=\varrho_1\varrho_2\cdots\varrho_s$$

* We note that throughout this paper the notion of "irreducible" and "prime" will be distinguished in the following sense: an $E(d)$-integer (different from zero and units) will be called irreducible if it has only trivial divisors, and prime if it cannot divide a product unless it divides at least one of the factors.

where π_ν and ϱ_μ are irreducible and $\pi_\nu \neq \varepsilon\,\varrho_\mu$ (ε is a unit), $r, s \geq 2$, I constructed an $\alpha_0 \in E(d)$ which had a smaller norm than α had and had also at least two different decompositions into irreducible factors. This process clearly leads to a contradiction.

We may assume

$$(2) \qquad |\pi_1| \leq |\varrho_1| \leq |\varrho_2| \leq \ldots \leq |\varrho_s| \,.$$

Not willing either to repeat the main ideas or to sketch the unaltered parts of the proof, I turn to that point when after several steps I found that it was enough to deal with the case in which

$$(3) \qquad |\varrho_2 \ldots \varrho_s| \leq l_d^* \leq \sqrt{2}\,\frac{\sqrt{-d}}{2} + 1 \,.$$

This was in the middle of page 462 of my paper. The next four pages of the argument can be replaced by the following consideration, using the above mentioned proposition.

It was previously established (p. 461), that the norms of all non-real irreducible $E(d)$-integers inside the circle of radius l_d^* are rational primes. Hence all these integers are primes in $E(d)$.

Combining (2) and (3) we obtain that $|\pi_1| \leq l_d^*$ and $|\varrho_\mu| \leq l_d^*$ for all μ. On the other hand none of the π_ν and ϱ_μ can be primes, since then we could reduce both sides of (1) by them. Thus $\pi_1, \varrho_1, \ldots, \varrho_s$ must be all real, i.e. they are rational primes. But then α is a rational integer and using that the rational integers form a UFD we obtain obviously $\pi_1 = \varepsilon\,\varrho_\mu$ for some μ, which is a contradiction.

We note that from our proof we get also the following result:

If for some $d < 0$ the norms of all non-real irreducible $E(d)$-integers inside the circle of radius $\frac{\sqrt{-d}}{\sqrt{2}} + 1$ are rational primes, then $E(d)$ is a UFD. This is a somewhat weaker form of a theorem of NAGELL in [2] (not known by me when writing [1]) obtained by completely different methods.

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A NOTE ON UNIQUE FACTORIZATION IN IMAGINARY QUADRATIC FIELDS

By

T. ZAUPPER

College of Transportation and Telecommunication, Győr

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Let $E(d)$ denote the ring of integers of the algebraic number field $Q(\sqrt{d})$ (the extension of the rational field by $\sqrt{d}$), where d is a rational integer. We may assume that

$$d = 4m+1$$

or

$$d = 4j \quad \text{where} \quad j \equiv 2 \quad \text{or} \quad 3 \pmod 4.$$

In [2] EDIT GYARMATI gave an elementary proof that for

$$d = -3,\ -4,\ -7,\ -8,\ -11,\ -19,\ -43,\ -67 \quad \text{and} \quad -163$$

$E(d)$ is a unique factorization domain (UFD) — see also [3].

Her method yielded also a result of the following type: if for some $d<0$, all $E(d)$-integers of absolute value $\le C_d$ (explicitly given) have a unique factorization, then $E(d)$ is a UFD.

Generalizing her ideas I obtained similar results for real quadratic fields (see [7]).

In this paper I shall sharpen the results of E. GYARMATI and also some of my results contained in [7], further I shall give an entirely new proof of an old theorem of T. NAGELL [4]. All these results are established by a careful analysis of E. GYARMATI's original proof and by several simplifications due to a lemma of C. P. POPOVICI [5] (see the Lemma below).

THEOREM 1. *Let be $d<0$, and $S = \sqrt{\frac{-d}{3}} \sim 0{,}58\sqrt{-d}$. Then $E(d)$ is a* UFD *if and only if the following condition holds:*

(1) *each real irreducible element p of $E(d)$ with $0<p\le S$ is also a prime in $E(d)$, i.e. it cannot divide a product unless it divides at least one of the factors.*

REMARKS.

1. We shall use the above distinction between "irreducible" and "prime" throughout the paper.

2. The necessity of condition (I) is obvious.

3. In [7] I proved a similar result with

$$T = \begin{cases} \dfrac{4\sqrt{-d}}{\pi} \sim 1{,}27\sqrt{-d} & \text{for} \quad d = 4m+1, \\ \dfrac{2\sqrt{-d}}{\pi} \sim 0{,}64\sqrt{-d} & \text{for} \quad d = 4j \end{cases}$$

instead of S.

THEOREM 2. *Let be $d<0$, $d = -4k+1$. Then the following three conditions are equivalent:*

(i) *$E(d)$ is a* UFD.
(ii) *n^2+n+k is a rational prime for all $0 \le n < k-1$.*
(iii) *n^2+n+k is a rational prime for all*

$$0 \le n \le \frac{S}{2} = \frac{1}{2}\sqrt{\frac{4k-1}{3}}.$$

REMARKS. 1. Theorem 2 is essentially equivalent to Theorem VIII of [4], but there the value of the upper bound in (iii) is

$$\frac{1}{2}\sqrt{\frac{4k+15}{3}} - \frac{5}{2}$$

and also $d < -59$ is assumed, further the proof is completely different.

2. We shall prove only the implication (iii)⇒(i). (i)⇒(ii) is easy, and can be found e.g. in [4], and (ii)⇒(iii) is obvious.

3. In [1] BEEGER claims that $n^2+n+72491$ is prime for all $0 \le n \le 11000$. By Theorem 2 this must be false, since for $d = (-4)\cdot 72491+1$ $E(d)$ is not UFD (no $E(d)$ is a UFD for $d < -163$, see [6]).

PROOF OF THE THEOREMS. The basic ideas and also several details of the construction are the same as those of E. GYARMATI. To make this paper self-contained we repeat now briefly those steps too, which are taken from [2] unaltered.

The major part of the proofs of the two theorems runs together.

Assume indirectly that $E(d)$ is not a UFD, and α has the minimal norm among the numbers, which have at least two distinct decompositions.

Then

$$\alpha = \pi_1 \pi_2 \dots \pi_r = \varrho_1 \varrho_2 \dots \varrho_s$$

holds where π_ν and ϱ_μ are irreducible, $r, s \ge 2$ and $\pi_\nu \ne \varepsilon \varrho_\mu$ (ε is a unit in $E(d)$). Clearly π_ν and ϱ_μ cannot be primes in $E(d)$. Assume that

$$N(\pi_1) \le N(\varrho_1) \le N(\varrho_2) \le \dots \le N(\varrho_s).$$

Let us consider

$$\alpha_0 = \alpha\,\xi - \frac{\pi_1}{\varrho_1}\alpha\,\eta = \pi_1(\pi_2 \ldots \pi_r\,\xi - \varrho_2 \ldots \varrho_s\,\eta) = \varrho_2 \ldots \varrho_s(\varrho_1\,\xi - \pi_1\,\eta)$$

where the non-zero $E(d)$-integers ξ and η will be chosen suitably later. If

$$\pi_1 \nmid \varrho_1\,\xi \tag{2}$$

then α_0 will also have at least two different factorizations (one with π_1 and one without π_1), which is a contradiction, if also $N(\alpha_0) < N(\alpha)$, i.e.

$$\left|\xi - \frac{\pi_1}{\varrho_1}\eta\right| < 1 \tag{3}$$

holds. We show that both (2) and (3) can be satisfied by a suitable choice of ξ and η, and this will complete the proof.

First we deal with condition (3). Since

$$N(\pi_1) \le N(\varrho_1), \quad \left|\frac{\pi_1}{\varrho_1}\right| \le 1\,.$$

If the point $\frac{\pi_1}{\varrho_1}$ lies in the interior of the unit circle around the point 1 or -1, then (3) holds with the choice $\xi = 1$ or -1, and $\eta = 1$. In this case (2) is clearly valid too. Thus we may assume that $\frac{\pi_1}{\varrho_1}$ belongs to the shaded domain in Figure 1.

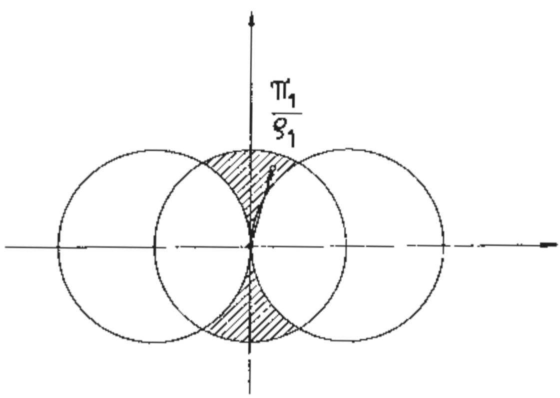

Fig. 1.

Then $\frac{\pi_1}{\varrho_1}$ is a point of a line e, which contains the origin, and its angle with the positive part of the real axis is between 60° and 120°. Let us denote the extremal positions, of the line e by e_1 and e_2, and let f be horizontal line with $y=\frac{\sqrt{-d}}{2}$, and finally consider the intersection point M of the lines e and f. (Line f is the first parallel line to the real axis which contains lattice-points, i.e. points of $E(d)$.)

We take now for ξ that lattice-point P of which is inside or on the border of the angle of 60° determined by e_1 and e_2, and is closest to M (see Figure 2).

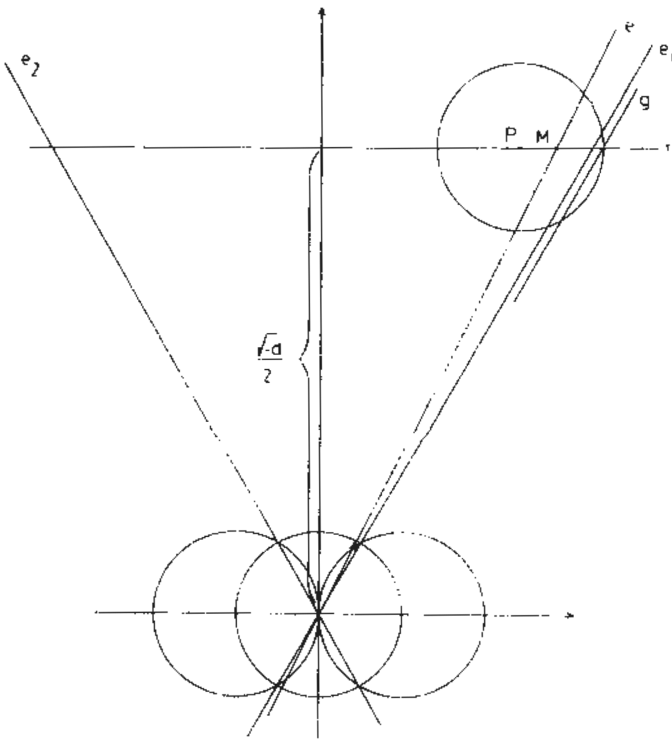

Fig. 2.

We verify that the distance of P and e is less than $\frac{\sqrt{3}}{2}$. This is obvious if P is not one of the extreme lattice-points of the section between e_1 and e_2 of f, since then

$$\overline{PM}\leq\frac{1}{2}$$

(the distance of the consecutive lattice-points on f is 1). If P is an extreme point, then consider the line g parallel to e_1, and having a distance $\frac{\sqrt{3}}{2}$ from P, the distance of e and P is obviously smaller (see Figure 2).

The multiples of $\frac{\pi_1}{\varrho_1}$ by consecutive rational integers are on e, and have a distance $\left|\frac{\pi_1}{\varrho_1}\right| < 1$. Since the distance of e and P is less than $\frac{\sqrt{3}}{2}$, therefore the unit circle around P contains in its interior a segment of e longer than 1, and so this segment must contain a multiple (by a rational integer) of $\frac{\pi_1}{\varrho_1}$. If we take this rational integer for η, then (3) is satisfied.

It is obvious that in this case

$$|\xi| \leq \sqrt{\frac{-d}{3}} = S$$

(see Figure 3).

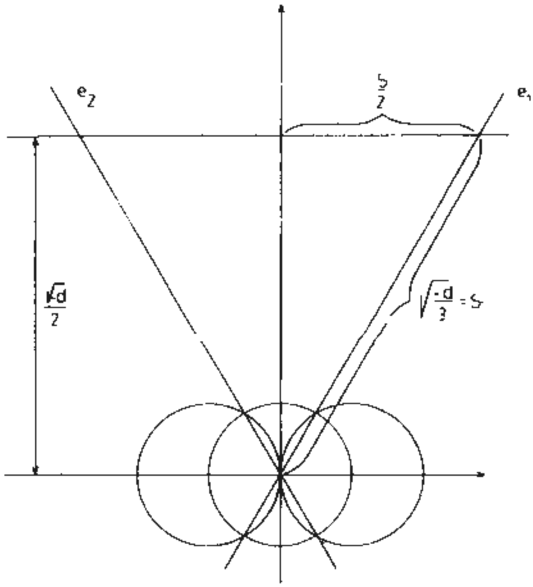

Fig. 3.

Now we turn to condition (2). We shall show that if ξ is prime or a product of primes, then (2) is valid. Assume indirectly, that for some β

$$\pi_1 \beta = \varrho_1 \xi .$$

Since ξ is prime or a product of primes, we can reduce here both sides by all factors of ξ. Hence we obtain

$$\pi_1 \beta' = \varrho_1$$

(π_1 must remain since it is not a prime), which is clearly a contradiction. Thus to prove our theorems it suffices to show that our ξ is prime or a product of primes.

In this final step we shall make frequently use of the following.

LEMMA. If $N(\gamma)=p$, where p is a rational prime, then γ is not only irreducible but it is also prime.

This lemma can be easily proved by ideal theory. An elementary but complicated proof was given in [5]. For a simple elementary proof see [7].

Let us turn now to Theorem 1. Assume again indirectly that ξ has an irreducible divisor δ which is not a prime.

Consider

$$\delta\bar{\delta}=N(\delta)=m.$$

Since

$$|\delta|\leq|\xi|\leq S$$

therefore

$$m\leq S^2.$$

m cannot be a rational prime by the Lemma.

Hence it has a rational prime divisor

$$p\leq\sqrt{m}\leq S.$$

If p is irreducible in $E(d)$, then by condition (1), it is also prime in $E(d)$. If p is reducible, then by the Lemma it is the product of two primes. Thus in any case p has a prime divisor π ($p=\pi$ or $p=\pi\bar{\pi}$), hence

$$\pi|m=\delta\bar{\delta}.$$

But then

$$\pi|\delta \quad \text{or} \quad \pi|\bar{\delta}$$

which is possible only if $\delta=\varepsilon\pi$ or $\delta=\varepsilon\bar{\pi}$ (where ε is a unit). Hence δ is a prime which is a contradiction. To prove Theorem 2 we have only to observe that

$$n^2+n+k=N\left(n+\frac{1+\sqrt{d}}{2}\right)$$

and our ξ is of the form

$$\xi=n+\frac{1+\sqrt{d}}{2}$$

with

$$|n|\leq\frac{S}{2}.$$

By (iii) we have that $N(\xi)$ is a rational prime, and we obtain by the Lemma that ξ is prime in $E(d)$.

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ON JENKINS RADICAL AND SEMISIMPLE RINGS

By

N. V. LOI

Department of Algebra and Number Theory, L. Eötvös University, Budapest

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1. Introductions

In this paper we shall deal with rings without small ideals and rings without essential ideals. In section 2 the radical theoretic aspects of such rings will be investigated and we shall give two characterizations of the Jenkins radical in terms of the above mentioned rings. The structure of certain Jenkins semisimple rings will be described in section 3.

Next, we shall recall some notions which we shall use in the sequel. A class $\mathbf{R}$ of rings is called *hereditary*, whenever $I \lhd A \in \mathbf{R}$ implies $I \in \mathbf{R}$, further, $\mathbf{R}$ is said to be *closed under subdirect sums*, if any subdirect sum of $\mathbf{R}$-rings is again in $\mathbf{R}$. We say that $\mathbf{R}$ is *closed under extensions*, if $B \lhd A$, $B \in \mathbf{R}$ and $A/B \in \mathbf{R}$ imply $A \in \mathbf{R}$. If $\mathbf{R}$ has the property that for any ascending chain $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_\gamma \subseteq \ldots$ of ideals of any ring A with $B_\gamma \in \mathbf{R}$ for each γ, it follows $\cup B_\gamma \in \mathbf{R}$, then we say that $\mathbf{R}$ has the *inductive property*. A semisimple class is a class of rings consisting of rings having 0-radical with respect to an appropriate *Kurosh-Amitsur radical*. Let $\mathbf{R}$ be any Kurosh-Amitsur radical class and $\mathcal{S}\mathbf{R}$ its semisimple class. If $\mathbf{S}$ is a hereditary class then $\mathcal{U}\mathbf{S}$ denotes the class of rings having no non-zero homomorphic image in $\mathbf{S}$. As is well-known, $\mathcal{U}\mathbf{S}$ is a Kurosh-Amitsur radical class (see, e.g. [10] Theorem 7.2.).

An ideal I of a ring R is called *small (essential)* in case if $I+J = R$ $(I \cap J = 0)$ implies $J = R$ $(J = 0)$ respectively. Essential ideals are also called large ideals.

PROPOSITION 1.1. *When I is small in R then in any image $\overline{R}$ of R the image of I is small.*

For the proof we refer to KASCH [2] (5.13. lemma (c)), or MICHLER [7] (Hilfssatz 1.3.).

It is known (ARMENDARIZ [1]) that the radical $\mathbf{R}$ is hereditary if and only if the semisimple class $\mathcal{S}\mathbf{R}$ is *closed under essential extensions*, that is, if A is a essential ideal of the ring R and $A \in \mathcal{S}\mathbf{R}$ imply $R \in \mathcal{S}\mathbf{R}$.

2. On the Jenkins Radical Class

Let **S** denote the class of all rings A such that (i) A is semiprime and (ii) A has no non-zero small ideal. By definition *every simple idempotent ring belongs to the class* **S**, *but* **S** *does not contain subdirectly irreducible rings which are not simple idempotent ones.*

Proposition 2.1. *The class* **S** *is hereditary.*

Proof. Suppose that $0 \neq K \triangleleft I \triangleleft A \in \mathbf{S}$ and let $\overline{K}$ denote the ideal of A generated by K. By the assumption A has no non-zero nilpotent ideal, so $\overline{K}^3 \neq 0$ and in view of Andrunakievich Lemma (see, e.g. [10] Lemma 12.1) we have $\overline{K}^3 \subseteq K$. Also $A \in \mathbf{S}$ implies that $\overline{K}^3$ is not small in A, therefore there exists an ideal $H \neq A$ of A such that $\overline{K}^3 + H = A$. Since $\overline{K}^3 \subseteq I$, by the modular law we have

$$I = (H + \overline{K}^3) \cap I \subseteq \overline{K}^3 + (H \cap I) \subseteq K + (H \cap I).$$

Thus $I = K + (H \cap I)$ and since $H \neq A$ it follows that $\overline{K}^3 \not\subseteq H$, so $I \not\subseteq H$, that is $H \cap I \neq I$. Thus K is not small in I. Since being semiprime is a hereditary property, the assertion is proved.

Proposition 2.2. *The class* **S** *is subdirectly closed.*

The proof is straightforward in view of Propodition 1.1 and of the fact that any subdirect sum of semiprime rings is again semiprime.

Since the Brown-McCoy semisimple rings are subdirect sums of simple rings with identity, Proposition 2.2 yields immediately

Corollary 2.3. **S** *contains every Brown-McCoy semisimple ring.*

The class **S**, however, is not a semisimple class because it fails to be closed under extensions.

Proposition 2.4. **S** *is not closed under extensions and not even under essential extensions.*

Proof. Let A be any simple idempotent ring without identity. By definition $A \in \mathbf{S}$ holds. Furthermore, A is an F-algebra for either F = rationals if A has characteristic 0 or $F = Z_p$ if A has characteristic p. Then for the split extension $R = A^*$ we have $A \in \mathbf{S}$ and $R/A \cong F \in \mathbf{S}$. But R is well-known to be subdirectly irreducible so $R \notin \mathbf{S}$.

In fact R is an essential extension of A so this proves the second assertion.

Though **S** is not a semisimple class, the class $\mathcal{U}\mathbf{S}$ is certainly a radical class as **S** is hereditary.

Next, let us consider the class **T** of all rings, which have no essential ideal. Obviously **T** contains all semisimple artinian rings and also all simple rings including the simple zero-rings. A reformulation of Lemma 2 of Olson and Jenkins [8] gives

Proposition 2.5. *The class* **T** *is homomorphically closed.*

PROPOSITION 2.6. *The class* $\mathbf{T}$ *is hereditary.*

PROOF. We shall make use of Lemma 1 of OLSON and JENKINS [8] which states the following: *If I is a non-zero ideal of a ring A then either I is a direct summand of A or there is an essential ideal of A containing I.* Now let I be an ideal of $A\in\mathbf{T}$. By the above result we have $A=I\oplus J$. If K is any ideal of I, then by the direct decomposition of A also $K\lhd A$ holds. Since $A\in\mathbf{T}$, we get $A=K\oplus L$ proving that K is not essential in A. Taking into consideration that I is a direct summand of A, it follows that every ideal of I is an ideal of A, too. Hence K is not essential in I and so $I\in\mathbf{T}$ follows.

PROPOSITION 2.7. *The class* $\mathbf{T}$ *has the inductive property.*

PROOF. Let $I_1\subseteq\ldots\subseteq I_\alpha\subseteq\ldots$ be an ascending chain of ideals of a ring A such that each I_α is in $\mathbf{T}$. Without loss of generality we may confine ourselves to the case $A=\bigcup I_\alpha$. Let K be any proper ideal of A. Now there is an index γ such that $K_\gamma=K\cap I_\gamma\neq I_\gamma$. Since $K_\gamma\lhd I_\gamma\in\mathbf{T}$, by [8] lemma 1 it follows $I_\gamma=K_\gamma\oplus L_\gamma$ and $L_\gamma\neq 0$. We claim that $L_\gamma\lhd A$. To see this, let us take an element $a\in A=\bigcup I_\alpha$, now there is an index β such that $a\in I_\alpha$. If $\beta\leq\gamma$ then $a\in I_\beta\subseteq I_\gamma$, therefore $a\cdot L_\gamma\subseteq I_\beta\cdot L_\gamma\subseteq L_\gamma$ and $L_\gamma\cdot a\subseteq L_\gamma$ hold.

If $\beta>\gamma$, then $I_\gamma\lhd I_\beta\in\mathbf{T}$ which implies a direct decomposition $I_\beta=I_\gamma\oplus H$ yielding $I_\beta=K_\gamma\oplus L_\gamma\oplus H$. Hence $L_\gamma\lhd I_\beta$ and so $aL_\gamma\subseteq I_\beta L_\gamma\subseteq L_\gamma$ and also $L_\gamma a\subseteq L_\gamma$ holds. Thus $L_\gamma\lhd A$, since $L_\gamma\subseteq I_\gamma$ we have

$$L_\gamma\cap K\subseteq L_\gamma\cap K\cap I_\gamma=L_\gamma\cap K_\gamma=0$$

and hence K is not essential in A, proving $A\in\mathbf{T}$.

The class $\mathbf{T}$ *is not subdirectly closed* as it is obvious from the following example. Let us consider the complete direct sum A of infinite simple rings A_α. Since the direct sum of the rings A_α is an essential ideal of A, it follows $A\notin\mathbf{T}$, though $A_\alpha\in\mathbf{T}$ for each α.

Similarly as in the proof of Proposition 2.4, one can see that $\mathbf{T}$ *is not closed under extensions.*

As it was already mentioned the class $\mathbf{T}$ contains beside the semisimple artinian rings also simple zero-rings. It seems to be useful to exclude the zero-rings by considering the class $\mathbf{T}_0$ consisting of all semiprime rings $A\in\mathbf{T}$. By [8] lemma 1 it is obvious that $\mathbf{T}_0\subseteq\mathbf{S}$.

PROPOSITION 2.8. *A ring $A\in\mathbf{T}$ belongs to $\mathbf{T}_0$ if and only if A is idempotent, moreover*

$$\mathbf{T}_0=\{A\in\mathbf{T}: A\ \textit{is hereditarily idempotent}\}.$$

PROOF. Assume that A is idempotent and let $I\neq 0$ be any ideal of A. Since $A\in\mathbf{T}$, [8] Lemma 1 yields $A=I\oplus K$ and by $A^2=A$, it follows: $I^2=I$ that is, A is hereditarily idempotent, and consequently semiprime.

Conversely, suppose that A is semiprime. For A^2 we have $A=A^2\oplus B$ and $B^2\subseteq A^2\cap B=0$. Since A is semiprime it follows $B=0$, that is $A^2=A$. The last assertion has been shown in the first part of the proof.

PROPOSITION 2.9. $\mathbf{T}_0$ *has the following properties*

(a) $\mathbf{T}_0$ *is homomorphically closed,*
(b) $\mathbf{T}_0$ *is hereditary,*
(c) $\mathbf{T}_0$ *has the inductive property.*

PROOF. (a) follows from Propositions 2.5 and 2.8, meanwhile (b) is trivial by Proposition 2.6.

It is well-known that the class $\mathbf{H}$ of all hereditarily idempotent rings has the inductive property, since it is a radical class. By $\mathbf{T}_0 = \mathbf{T} \cap \mathbf{H}$ the assertion (c) follows.

In view of the above proved properties, the classes $\mathbf{T}$ and $\mathbf{T}_0$ are close to those of hereditary radicals (though the extension property is missing). Since among the hereditarily idempotent rings the relation $B \triangleleft A$ is transitive, the lower radical $\mathcal{L}\mathbf{T}_0$ of the class $\mathbf{T}_0$ is given as $\mathcal{L}\mathbf{T}_0 = \{A$: every non-zero homomorphic image of a A has a non-zero ideal in $\mathbf{T}_0\}$.

Furthermore, $\mathcal{L}\mathbf{T}_0$ is a subidempotent radical. In spite of $\mathbf{T}_0 \subseteq \mathbf{S} \subseteq \mathcal{S}\mathcal{U}\mathbf{S}$. We have

PROPOSITION 2.10. $\mathcal{L}\mathbf{T}_0 \not\subseteq \mathcal{S}\mathcal{U}\mathbf{S}$ *and* $\mathbf{S} \not\subseteq \mathcal{L}\mathbf{T}_0$.

PROOF. Let us consider the class $\mathbf{K}$ of rings A each of which is isomorphic to every non-zero homomorphic image of it. As LEAVITT and VAN LEEUWEN [5] have shown the class $\mathbf{K}$ has the following properties

(i) *Every ring $A \in \mathbf{K}$ is subdirectly irreducible.*
(ii) *Every simple primitive ring B with minimal one-sided ideals and without identity, can be embedded as a proper ideal in a ring $A \in \mathbf{K}$.*

Hence there exists a ring $A \in \mathbf{K}$ which has an idempotent heart. By the definition of $\mathcal{L}\mathbf{T}_0$ we have $A \in \mathcal{L}\mathbf{T}_0$ and the heart of A is in $\mathbf{T}_0$. Since the heart of A is a small ideal, it follows $A \notin \mathbf{S}$. Also every non-zero homomorphic image of A is not in $\mathbf{S}$, that is $A \notin \mathcal{S}\mathcal{U}\mathbf{S}$, because A is in $\mathbf{K}$. On the other hand the ring Z of integers is in $\mathbf{S}$, but no non-zero ideal of Z is hereditarily idempotent and therefore $Z \notin \mathcal{L}\mathbf{T}_0$.

THEOREM 2.11. *If* $\mathbf{M}$ *denotes the class of all simple idempotent rings then* $\mathcal{U}\mathbf{M} = \mathcal{U}\mathbf{T}_0 = \mathcal{U}\mathbf{S}$ *though* $\mathbf{M} \subsetneq \mathbf{T}_0 \subsetneq \mathbf{S}$, $\mathbf{M} \neq \mathbf{T}_0 \neq \mathbf{S}$.

PROOF. The assertions $\mathbf{M} \subsetneq \mathbf{T}_0 \subset \mathbf{S}$ and $\mathbf{M} \neq \mathbf{T}_0 \neq \mathbf{S}$ are obvious. In view of $\mathbf{M} \subset \mathbf{T}_0 \subset \mathbf{S}$ we get $\mathcal{U}\mathbf{S} \subseteq \mathcal{U}\mathbf{T}_0 \subseteq \mathcal{U}\mathbf{M}$. Let us consider a ring $A \notin \mathcal{U}\mathbf{S}$ and let us assume that $A \in \mathcal{U}\mathbf{M}$. Then there is a homomorphic image $B \neq 0$ of A such that $B \in \mathbf{S}$. By MICHLER [7] Hilfssatz 3.1 B has maximal ideals. If $B^2 = B$, then B as well as A has a non-zero homomorphic image in $\mathbf{M}$, and hence $A \notin \mathcal{U}\mathbf{M}$. If $B^2 \neq B$, then by $B \in \mathbf{S}$ we have $B^2 \neq 0$ and B^2 is not small in B, therefore, there exists a maximal ideal C of B such that $B^2 + C = B$. Now B/C is an idempotent simple ring, which is in $\mathbf{M}$. Since B/C is a homomorphic image of A, it follows $A \notin \mathcal{U}\mathbf{M}$.

JENKINS called an ideal I of a ring A a *special ideal*, if $A^n \subseteq I$ for some integer $n \geq 2$. If no such integer exists for I, then I is called *non-special*. A ring A is said to be an m-ring if A has no non-special maximal ideals. The class of all m-rings is a radical class, which is called the *Jenkins radical*. Using a result of DE LA ROSA [9] VAN LEEUWEN proved that the Jenkins radical class coincides with $\mathcal{U}\mathbf{M}$. OLSON and JENKINS characterized $\mathcal{U}\mathbf{M}$ as the class of all rings A such that every non-zero homomorphic image of A contains either an essential ideal or an ideal isomorphic to a simple zero-ring. Now Theorem 2.11 gives two new characterizations of the Jenkins radical. Relationship between the Jenkins radical and the other radicals is given in [6] by VAN LEEUWEN. Moreover, the Jenkins radical is not hereditary (cf. LEAVITT—JENKINS [4]).

3. On Jenkins semisimple rings

In this section we shall investigate the structure of certain Jenkins semisimple rings. For this purpose we introduce the operator ϱ designating to any ideal I of a ring A the ideal

$$\varrho(I) = \bigcap_{\alpha} (K_\alpha \lhd A : I + K_\alpha = A).$$

PROPOSITION 3.1. *$J \subseteq I$ implies $\varrho(I) \subseteq \varrho(J)$, for any ideal I and J of a ring A. I is a small ideal in A if and only if $\varrho(I) = A$.*

The assertions are immediate consequences of the definition of ϱ.

PROPOSITION 3.2. *For any ideal I of a ring A the ideals $I \cap \varrho(I)$ and $\varrho(I + \varrho(I))$ are small in A.*

PROOF. If K is an ideal of A such that $(I \cap \varrho(I)) + K = A$, then also $I + K = A$ holds. Hence $\varrho(I) \subseteq K$ is valid implying $I \cap \varrho(I) \subseteq K$. Thus $K = A$, proving the first assertion.

By Proposition 3.1 we have $\varrho(I + \varrho(I)) \subseteq \varrho(I)$ and $\varrho(I + \varrho(I)) \subseteq \varrho^2(I)$. Since by Proposition 3.2 $\varrho(I) \cap \varrho^2(I)$ is small in A, by

$$\varrho(I + \varrho(I)) \subseteq (\varrho(I) \cap \varrho^2(I))$$

it follows the second assertion.

Proposition 3.1 and 3.2 yield the following

PROPOSITION 3.3. *A semiprime ring A is in the class $\mathbf{S}$ if any only if $I \cap \varrho(I) = 0$ for any $I < A$.*

PROPOSITION 3.4. *Let A be a ring in $\mathbf{S}$. A is prime if and only if $\varrho(I) = 0$ for every non-zero ideal I of A.*

PROOF. If A is prime, then by Proposition 3.3 we have $\varrho(I)\, I \subseteq I \cap \varrho(I) = 0$. Since A is prime, it follows either $I = 0$ or $\varrho(I) = 0$.

Next suppose that $\varrho(I) = 0$ for every non-zero ideal I of A, and let $J \neq 0$ and $K \neq 0$ be ideals of A such that $J \cdot K = 0$. Since $\mathbf{S}$ consists of semiprime rings, $A \in \mathbf{S}$ implies $K^2 \neq 0$. Let us consider the ideals H_α such that $J + H_\alpha = A$. For any H_α we have $K \subseteq J + H_\alpha$ and so

$$0 \neq K^2 \subseteq (J + H_\alpha)\, K = J \cdot K + H_\alpha \cdot K \subseteq H.$$

This implies $0 \neq K^2 \subseteq \varrho(J)$ contradicting the hypothesis. Thus A is a prime ring.

Let Soc A denote the socle of A which is the sum of all minimal two sided ideals of A.

PROPOSITION 3.5. *If a prime ring A is in* **S**, *then either A is simple or* Soc $A=0$.

PROOF. Since A is prime, either Soc $A=0$ or Soc A is the unique minimal ideal of A. Since A is prime, Soc A must not be a direct summand of A and so either Soc $A=0$ or Soc A is the smallest ideal of A, and in this case A is subdirectly irreducible with heart Soc A. By the definition of **S**, however, **S** does not contain subdirectly irreducible rings, which are not simple idempotent rings. Thus either Soc $A=0$ or A is simple.

Let us recall that in view of Theorem 2.11 $\mathcal{S}\mathcal{U}\mathbf{S}$ is the class of all *Jenkins semisimple* rings. The next theorem characterizes those Jenkins semisimple rings A which are in $\mathbf{T}_0$.

THEOREM 3.6. *For a ring A the following conditions are equivalent*

(a) *A is semiprime and $\varrho^2(I)=I$ for every $I\lhd A$.*
(b) *A is semiprime and for every ideal I of A there exists a positive integer n such that $\varrho^{2n}(I)=I$.*
(c) *$A\in\mathbf{S}$ and $A=I+\varrho(I)$ holds for every $I\lhd A$.*
(d) *$A\in\mathbf{T}_0$.*
(e) *A is semiprime and each ideal of A is a direct summand of A.*
(f) *A is the direct sum of simple idempotent rings.*

We remark that the equivalence of (d), (e) and (f) is known, but for the sake of completeness we sketch also the proof of (e)⇒(f).

PROOF. (a)⇒(b). Clear.

(b)⇒(c). Let J be a small ideal of A. Then by definition $\varrho(J)=A$ and hence $\varrho^2(J)=\varrho(A)=0$. By Proposition 3.1 we get clearly $\varrho^{2n}(J)=0$ for all positive integers n. Thus (b) implies $J=0$, that is A has no non-zero small ideals. Hence $A\in\mathbf{S}$.

Next let I be any ideal of A. Since $A\in\mathbf{S}$, Proposition 3.2 implies $\varrho\big(I+\varrho(I)\big)=0$. Hence it follows $\varrho^2\big(I+\varrho(I)\big)=A$ and also $\varrho^{2n}\big(I+\varrho(I)\big)=A$ for all n. Thus (b) yields $I+\varrho(I)=A$.

(c)⇒(d). Let I be any proper ideal of A. By Proposition 3.2 $I\cap\varrho(I)$ is small in A and we have $I+\varrho(I)=A$ and therefore $\varrho(I)\neq 0$. Thus I is not essential in A.

(d)⇒(e). The implication follows from Olson—Jenkins [9] Theorem 3.

(e)⇒(f). By Birkhoff's Theorem A is a subdirect sum of subdirectly irreducible rings A_α $(\alpha\in\Lambda)$. Denoting by π_α the projection of A onto A_α, (e) yields the decomposition $A=\ker\pi_\alpha\oplus A_\alpha$. The heart of A_α is an ideal of A as well hence a direct summand of A and also of A_α. Hence necessarily $A_\alpha=H(A_\alpha)$, that is A_α is simple. Since A is semiprime, A_α is idempotent. Let us consider the socle Soc A of A which is the sum of all simple idempotent rings $A_\alpha\lhd A$. It can be verified that Soc A is the direct sum of the rings A_α (the proof is the same as in the case of modules (cf. LAMBEK [3] p. 59—60)). We prove that $A=$ Soc A. Otherwise by condition (e) we have $A=$ Soc $A\oplus B$ $(B\neq 0)$

and obviously also B satisfies condition (e). Hence B has an ideal C which is a simple idempotent ring. Since C is an ideal of A too, it follows $C \subseteq \operatorname{Soc} A$, and concequently $B \cap \operatorname{Soc} A \neq 0$, a contradiction, Hence $A = \operatorname{Soc} A$.

(f)$\rightarrow$(a). Let I be any ideal of A, and A be a direct sum of simple idempotent rings A_α, $\alpha \in \Lambda$. It is known that an ideal of a direct sum of simple idempotent rings is of the same kind. It follows that if $\Lambda' = \{\beta \in \Lambda : A_\beta \subseteq I\}$ then I is the direct sum of A_β, $\beta \in \Lambda'$. Hence $A = I \oplus J$, where J is a direct sum of A_γ, $\gamma \in \Lambda \setminus \Lambda'$.

Let K be any ideal of A such that $I + K = A$. We will show that $J \subseteq K$, and therefore $J = \varrho(I)$. Let $0 \neq b \in J$, then $b = a_1 + a_2 + \ldots + a_m$ and $b = b_1 + b_2$ where $0 \neq a_i \in A_i$, $i \in \Lambda \setminus \Lambda'$ and $b_1 \in I$, $b_2 \in K$. Since

$$a_i \in A_i = A_i a_i A_i = A_i b A_i = A_i b_2 A_i \subseteq K$$

we have $(a_1 + \ldots + a_m) \in K$. Thus $\varrho(I) = J$. Similarly, we can show that $\varrho(J) = I$. Hence $I = \varrho(J) = \varrho^2(I)$. Obviously A is also semiprime thus condition (a) holds. The proof of Theorem 3.6 is complete.

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MOMENT TYPE THEOREMS FOR *-SEMIGROUPS OF OPERATORS ON HILBERT SPACE I.

By

ZOLTÁN SEBESTYÉN

II. Department of Math. Analysis, L. Eötvös University, Budapest

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Introduction

Given a multiplicative semigroup G with unit e and involution *, briefly a *-semigroup, an operator valued function F of G into $B(H)$, the C^*-algebra of all bounded linear operators on the complex Hilbert space H, has a dilation S on a Hilbert space K (in a general sense) if there is a continuous linear map $V: K \to H$ with

$$F(g) = V S_g V^* \quad \forall g \in G \tag{1}$$

where $S: G \to B(K)$ is a *-representation of G on the Hilbert space K. In case when $F(e) = I_H$ (the identity operator on H), or equivalently $V V^* = I_H$ holds in (1), S is called a strong dilation of F on K.

Let further be given a vector valued function x of G into the Hilbert space H, then it seems to be natural to ask:

A. Under what condition does there exist a dilatable operator valued function F on G, such that

$$x_g = F(g) x_e \quad \forall g \in G \tag{2}$$

holds. Our aim is to answer this question. It should be remarked that this problem in case of a strong dilation is left open. Only a special but interesting case will be treated.

B. Given a sequence $\{x_n\}_{n=-\infty}^{\infty}$ of elements of H, which spans the space H, when does exist a contraction T on H such that

$$x_n = T_n x_0 \quad \forall n \in \mathbf{Z}, \tag{3}$$

where $T_n = T^n$ for $n \geqq 0$ and $T_n = T^{*|n|}$ for $n < 0$ and $\mathbf{Z}$ is the set of entire numbers.

C. A similar problem is solved for one parameter continuous group of contractions on H. These last two problems are continuations of our previous investigations in [2].

Results

THEOREM A. *Problem A has a solution if and only if there exist a constant $M\geqq 0$ and a submultiplicative function $p: G\to R^+$ with C*-property*

$$p(g^*g)=p(g)^2 \quad \forall g\in G, \tag{4}$$

$$\left(\textit{implying} \quad p(g^*)=p(g) \quad \forall g\in G\right)$$

such that

$$\|x_g\|\leqq M\,p(g) \quad \forall g\in G, \tag{5}$$

$$\Big\|\sum_g c_g x_g\Big\|^2\leqq M\sum_{g,h} c_g \bar c_h (x_{h^*g}, x_e) \tag{6}$$

holds for all finite sequence $\{c_g\}$ of complex numbers indexed by elements of G.

PROOF. Assuming a solution $F: G\to B(H)$ with (1) and (2) one has for any g in G

$$\|x_g\|\leqq\|F(g)\|\,\|x_e\|\leqq\|V\|\,\|S_g\|\,\|V^*\|\,\|x_e\|\leqq M\,p(g)$$

where $M=\|V\|^2\|x_e\|$, $p(g)=\|S_g\|$ is of the type desired in the statement; and also

$$\Big\|\sum_g c_g x_g\Big\|^2\leqq\|V\|^2\Big\|\sum_g c_g S_g V^* x_e\Big\|^2\leqq\|V\|^2\sum_{g,h} c_g c_h (V S_{h^*g} V^* x_e, x_e) =$$
$$=\|V\|^2\sum_{g,h} c_g c_h (x_{h^*g}, x_e)$$

for any finite sequence $\{c_g\}_{g\in G}$ of complex numbers.

On the other hand let Y be the linear space of all complex valued functions on G with finite support, each of which is of the form $\sum_g c_g\delta_g$, where δ_g denotes the function 1 in g and 0 otherwise and $\{c_g\}$ is a finite sequence of complex numbers indexed by elements of G. Note that $\sum$ denotes always a finite sum. For any two elements

$$\sum_h c_h\delta_h \quad\text{and}\quad \sum_k d_k\delta_k \quad\text{of}\quad Y$$

we define a complex number by

$$\Big\langle\sum_h c_h\delta_h, \sum_k d_k\delta_k\Big\rangle=\sum_{h,k} c_h d_k (x_{k^*h}, x_e)$$

such giving a semi-inner product on Y because of (6). The resulting Hilbert space (after factorization and completion) is K. In view of (6) we have also that the map V of Y into H given by

$$V\Big(\sum_h c_h\delta_h\Big):=\sum_h c_h x_h \quad \Big(\sum_h c_h\delta_h\in Y\Big) \tag{7}$$

defines a continuous linear map V of K into H (by a unique extension of the densely defined continuous linear map arising after factorization). We prove first the identity:

$$V^*x_e=\delta_e. \tag{8}$$

Let $\sum_h c_h \delta_h$ be an arbitrarily chosen element from Y, we have by (7)

$$\left\langle \sum_h c_h \delta_h, V^* x_e \right\rangle = \left(V \left(\sum_h c_h \delta_h\right), x_e\right) = \sum_h c_h (x_h, x_e) = \left\langle \sum_h c_h \delta_h, \delta_e \right\rangle$$

implying (8) since such elements are dense in K.

Consider now the shift operator S_g on Y given by

$$S_g \left(\sum_h c_h \delta_h\right) := \sum_h c_h \delta_{gh} \quad \left(g \in G, \ \sum_h c_h \delta_h \in Y\right). \tag{9}$$

We have to show that S_g is continuous. To see this we observe that for any $y = \sum_h c_h \delta_h$ in Y

$$\|S_g y\|^2 = \left\|\sum_h c_h \delta_{gh}\right\|^2 = \sum_{h,k} c_h \bar{c}_k (x_{k^* g^* gh}, x_e) = \langle S_{g^* g} y, y \rangle \leqq \|S_{g^* g} y\| \, \|y\| .$$

By induction we have in consequence of (5), (6)

$$\|S_g y\|^{2^{n+1}} \leqq \|S_{(g^* g)^{2^n - 1}} y\|^2 \cdot \|y\|^{2^{n+1}-2} =$$
$$= \|y\|^{2^{n+1}-2} \sum_{h,k} c_h \bar{c}_k (x_{k^* (g^* g)^{2^n} h}, x_e) \leqq$$
$$\leqq \|y\|^{2^{n+1}-2} M \|x_e\| \sum_{h,k} |c_h| \, |c_k| \, p\left(k^* (g^* g)^{2^n} h\right) \leqq$$
$$\leqq p(g)^{2^{n+1}} \|y\|^{2^{n+1}-2} M \|x_e\| \left(\sum_h |c_h| \, p(h)\right)^2 ,$$
$$\|S_g y\| \leqq p(g) \|y\|^{1-2^{-n}} \left\{M \|x_e\| \left(\sum_h |c_h| \, p(h)\right)^2\right\}^{2^{-n-1}}$$

for any $n = 0, 1, 2, \ldots$ and thus finally that

$$\|S_g y\| \leqq p(g) \|y\| \quad (g \in G; \ y \in Y).$$

We see, as in case of V, that S_g is a bounded linear operator on K as well. Moreover it is a simple check to prove that S, as an operator valued function on G, preserves multiplication and involution. In other words S is a *-representation of G on K. Now it is a kind duty to observe that the operator valued function F on G as in (1) satisfies (2) too: by (8), (9) and (7) we have

$$F(g) x_e = V S_g V^* x_e = V S_g \delta_e = V \delta_g = x_g$$

for any g in G indeed. The proof is complete.

THEOREM B. *Problem* B *has a solution if and only if there exists a double sequence* $\{x_n^{n'}\}_{n', n = -\infty}^{\infty}$ *of elements of* H *such that*

$$x_n^{n'} = x_{n' + n} \quad \textit{whenever} \quad n' \cdot n \geqq 0 \tag{10}$$

and

$$\left\|\sum_{n', n} c_{n', n} x_n^{n'}\right\|^2 \leqq \sum_{\substack{m', m \\ n', n}} c_{m', m} \bar{c}_{n', n} (x_m^{m'-n'}, x_n) \tag{11}$$

hold for any finite sequence $\{c_{n',n}\}$ of complex numbers indexed by pairs of entire numbers.

PROOF. Assume first that we have a contraction T on the Hilbert space H satisfying (3). Let $x_n^{n'} := T_{n'} x_n$ for any two entire numbers n', n. Then for $n', n \geqq 0$ we have

$$x_n^{n'} = T_{n'} T_n x_0 = T^{n'} T^n x_0 = T^{n'+n} x_0 = T_{n'+n} x_0 = x_{n'+n},$$

and for $n', n < 0$ that

$$x_n^{n'} = T^{*(-n')} T^{*(-n)} x_0 = T^{*|n'+n|} x_0 = T_{n'+n} x_0 = x_{n'+n}.$$

Moreover for any finite sequence $\{c_{n',n}\}_{n',n=-\infty}^{\infty}$ it follows by the unitary dilation of B. SZ.-NAGY that

$$\Big\|\sum_{n',n} c_{n',n} x_n^{n'}\Big\|^2 = \Big\|\sum_{n',n} T_{n'} x_n\Big\|^2 = \Big\|P \sum_{n',n} c_{n',n} U^{n'} x_n\Big\|^2 = \Big\|\sum_{n',n} c_{n',n} U^{n'} x_n\Big\|^2 =$$

$$= \sum_{\substack{m',m \\ n',n}} c_{m',m} \bar{c}_{n',n} (U^{m'-n'} x_m, x_n) = \sum_{\substack{m',m \\ n',n}} c_{m',m} \bar{c}_{n',n} (T_{m'-n'} x_m, x_n) =$$

$$= \sum_{\substack{m',m \\ n',n}} c_{m',m} \bar{c}_{n',n} (x_m^{m'-n'}, x_n),$$

where U is the unitary dilation of T on a Hilbert space K containing H. These show the necessity of (10) and (11). On the other hand, assuming (10) and (11), let Y be the linear space of complex valued functions with finite support on $\mathbf{Z} \times \mathbf{Z}$, each of which is of the form $\sum_{n',n} c_{n',n} \delta_n^{n'}$, where $\delta_n^{n'}$ denotes the function 1 in (n', n) and 0 otherwise and $\{c_{n',n}\}$ is a finite sequence of complex numbers indexed by pairs of entire numbers in $\mathbf{Z}$. For elements $\sum_{m',m} c_{m',m} \delta_m^{m'}$ and $\sum_{k',k} d_{k',k} \delta_k^{k'}$ of Y we associate a complex number by

$$\Big\langle \sum_{m',m} c_{m',m} \delta_m^{m'}, \sum_{k',k} d_{k',k} \delta_k^{k'} \Big\rangle := \sum_{\substack{m',m \\ k',k}} c_{m',m} \bar{d}_{k',k} (x_m^{m'-k}, x_k)$$

thus defining (by (11)) a semi-inner product on Y. Let K be the resulting Hilbert space, as before, a contractive linear map V of which into H exists (also by (11)) defining on Y for $\sum_{m',m} c_{m',m} \delta_m^{m'}$ as

$$V\Big(\sum_{m',m} c_{m',m} \delta_m^{m'}\Big) := \sum_{m',m} c_{m',m} x_m^{m'}. \tag{12}$$

Note that at once $VV^* = I_H$ follows because

$$V^* x_n = \delta_n^0 \quad \forall n \in \mathbf{Z} \tag{13}$$

holds for V since for any $y = \sum_{m',m} c_{m',m} \delta_m^{m'}$ in Y we have by (12)

$$\Big\langle \sum_{m',m} c_{m',m} \delta_m^{m'}, V^* x_n \Big\rangle = \Big(V\Big(\sum_{m',m} c_{m',m} \delta_m^{m'}\Big), x_n^0\Big) =$$

$$= \sum_{m',m} c_{m',m} (x_m^{m'}, x_n^0) = \Big\langle \sum_{m',m} c_{m',m} \delta_m^{m'}, \delta_n^0 \Big\rangle.$$

Now let S be the shift on Y given by

$$S\Big(\sum_{m',m} c_{m',m}\,\delta_m^{m'}\Big) := \sum_{m',m} c_{m',m}\,\delta_m^{m'+1}. \tag{14}$$

An easy calculation shows that we have indeed a unitary operator S on K, since $S^* = S^{-1}$ in view of

$$S^*\Big(\sum_{m',m} c_{m',m}\,\delta_m^{m'}\Big) = \sum_{m',m} c_{m',m}\,\delta_m^{m'-1} \tag{15}$$

and S is an isometry with respect to the norm inherited from the inner product given before on the factor space of Y.

Finally let $T = V S V^*$ be the contraction of H which is the claimed solution of the problem. We have indeed by (13), (14), (12) for $n \geqq 0$

$$T x_n = V S V^* x_n = V S \delta_n^0 = V \delta_n^1 = x_n^1 = x_{n+1}$$

and also by (15) for $n < 0$

$$T^* x_n = V S^* V^* x_n = V S^* \delta_n^0 = V \delta_n^{-1} = x_n^{-1} = x_{n-1}.$$

These show (3) and the proof is ended.

COROLLARY B. *Problem* B *has unitary operator as a solution if and only if*

$$(x_m, x_n) = (x_{m-n}, x_0) \quad \forall\, m, n \in \mathbf{Z} \tag{16}$$

PROOF. Let $x_n^{n'} := x_{n'+n}$, for any $n', n \in \mathbf{Z}$, then

$$\Big\|\sum_{n',n} c_{n',n} x_n^{n'}\Big\|^2 = \sum_{\substack{m',m\\ n',n}} c_{m',m}\bar c_{n',n}(x_m^{m'}, x_n^{n'}) = \sum_{\substack{m',m\\ n',n}} c_{m',m}\bar c_{n',n}(x_{m'+m}, x_{n'+n}) =$$

$$= \sum_{\substack{m',m\\ n',n}} c_{m',m}\bar c_{n',n}(x_{m'-m+n'-n}, x_0) = c_{m',m}\bar c_{n',n}(x_m^{m'-n'}, x_n),$$

hence equality holds in (11) for any finite sequence $\{c_{n',n}\}_{n',n=-\infty}^{\infty}$ of complex numbers. This implies that V is unitary in the proof of Theorem B and thus $T = V S V^*$ is unitary too.

The proof of the next theorem is the continuous counterpart of the proof of Theorem B using the suitable dilation theory of B. SZ.-NAGY.

THEOREM C. *Given a one parameter continuous family* $\{x_t\}_{t\in\mathbf{R}}$ *in* H *there exists a continuous family* $\{T_t\}_{t\in\mathbf{R}}$ *of contractions on* H *with* $T_{-t} = T_t^*$ *for* $t \in \mathbf{R}$ *and* $T_0 = I_H$, *for which*

$$x_t = T_t x_0 \quad \forall\, t \in \mathbf{R} \tag{17}$$

holds if and only if there exists a continuous family $\{x_t^{t'}\}_{t',t\in\mathbf{R}}$ *in* H *such that*

$$x_t^{t'} = x_{t'+t} \quad \textit{for} \quad t'\cdot t \geqq 0 \tag{18}$$

and

$$\Big\|\sum_{t',t} c_{t',t} x_t^{t'}\Big\|^2 \leqq \sum_{\substack{s',s\\ t',t}} c_{s',s}\bar c_{t',t}(x_s^{s'-t'}, x_t) \tag{19}$$

hold for any finite sequence $\{c_{t',t}\}$ of complex numbers indexed by pairs of real numbers.

COROLLARY C. Theorem C has a continuous family $\{U_t\}_{t\in\mathbf{R}}$ of unitaries on H with $U_{-t}=U_t^{-1}$ as a solution if and only if

$$(x_s, x_t) = (x_{s-t}, x_0) \quad \forall\, s, t \in \mathbf{R}.$$

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